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Set-Valued Mapping and Rough Probability

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Abstract: In 1982, the theory of rough sets proposed by Pawlak and in 2013, Luay concerned a rough probability by using the notion of Topology. In this paper, we study the rough probability in the stochastic approximation spaces by using set-valued mapping and obtain results on rough expectation, and rough variance.

Keywords: Rough set; Lower inverse approximation, Upper inverse approximation, Set-valued mapping, Stochastic approximation space, rough expectation, rough variance.

1 Introduction and Preliminaries

The theory of rough sets was first introduced by Pawlak [15]. Rough set theory, a new mathematical approach to deal with inexact, uncertain or vague knowledge, has recently received wide attention on the research areas in both of the real-life applications and the theory itself. Also, after the proposal by Pawlak, there have been many researches on the connection between rough sets and algebraic systems [1,2,3,4,5,6,7,9,10,11,12,13]. In [8] Jamal study stochastic approximation spaces from topological view that generalize the stochastic approximation space in the case of general relation. The couple $S=(X,P)$ is called the stochastic approximation space, where X is a non-empty set and P is a probability measure.

Lower and upper inverse is defined as follows:

Definition 1.1 Let X be a non-empty set and $A\subseteq X$. Let $T:X\longrightarrow P^*(X)$ be a set-valued mapping where $P^*(X)$ denotes the set of all non-empty subsets of X. The lower inverse and upper inverse of A under T are defined as

$$T^+(A)=\{x\in X|T(x)\subseteq A\}\ ;\ T^{-1}(A)=\{x\in X|T(x)\cap A\neq\emptyset\},$$

respectively. Also, $(T^+(A),T^{-1}(A))$ is called T-rough set of X.

Also, using lower and upper inverse, the lower and upper probability is defined as follows:

Definition 1.2 Let $T:X\longrightarrow P^*(X)$ be a set-valued mapping and A be an event in the stochastic approximation space $S=(X,P)$. Then the lower and upper probability of A is given by:

$$\underline{P}(A)=P(T^+(A))\ ;\ \overline{P}(A)=P(T^{-1}(A)),$$

respectively. Clearly, $0\leq\underline{P}(A)\leq 1$ and $0\leq\overline{P}(A)\leq 1$.

Definition 1.3 Let X be a non-empty set. Let $T:X\longrightarrow P^*(X)$ be a set-valued mapping. Then we say T has

i) reflective property, if for every $x\in X$ we have $x\in T(x)$,

ii) transitive property, if for every $y\in T(x)$ and $z\in T(y)$ we have $z\in T(x)$.

Remark 1.4 Let T has reflective and transitive properties, then in topological space (X,τ) we have $T^{-1}(A)=\overline{A}$ and $T^+(A)=A^o$, where A^o denotes interior of A and $\overline{A}$ denotes the closure of A. These implies that Definition **??** of our paper is same the Definition 2.2 of paper [14]. Hence this paper is generalized version of paper [14].

Definition 1.5 Let A be a subset of topological space (X,τ), then we call A is a exact set if $T^+(A)=T^{-1}(A)=A$.

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2 Main Result

Proposition 2.1 Let $T : X \longrightarrow P^*(X)$ be a set-valued mapping and A, B be two events in the stochastic approximation space $S = (X, P)$. Then the following holds:

(1)$\underline{P}(\emptyset) = 0 = \overline{P}(\emptyset)$;
(2)$\underline{P}(X) = 1 = \overline{P}(X)$;
(3)$\overline{P}(A \cup B) \leq \overline{P}(A) + \overline{P}(B) - \overline{P}(A \cap B)$;
(4)$\underline{P}(A \cup B) \geq \underline{P}(A) + \underline{P}(B) - \underline{P}(A \cap B)$;
(5)$\underline{P}(A^c) = 1 - \overline{P}(A)$;
(6)$\underline{P}(A - B) \leq \underline{P}(A) - \underline{P}(A \cap B)$;
(7)$\underline{P}(A) \leq \overline{P}(A)$;
(8)If $A \subseteq B$, then $\underline{P}(A) \leq \underline{P}(B)$ and $\overline{P}(A) \leq \overline{P}(B)$.

Proof. It is straightforward.

Definition 2.2 Let $T : X \longrightarrow P^*(X)$ be a set-valued mapping and A be an event in the stochastic approximation space $S = (X, P)$. The rough probability of A, denoted by $P^*(A)$, is given by:

$$P^*(A) = (\underline{P}(A), \overline{P}(A)).$$

Lemma 2.3 Let $T : X \longrightarrow P^*(X)$ be a set-valued mapping and A be a event in the stochastic approximation space $S = (X, P)$.

(1)If T has reflective property, then $\underline{P}(A) \leq P(A) \leq \overline{P}(A)$;
(2)If T has reflective and transitive properties, then $\underline{P}(T^+(A)) = \underline{P}(A)$ and $\overline{P}(T^{-1}(A)) = \overline{P}(A)$;
(3)If A is an exact subset of X, then $\underline{P}(A) = P(A) = \overline{P}(A)$.

Proof. It is straightforward.

Example 2.4 Let $X = \{1, 2, 3, 4, 5, 6\}$ and let $T : X \longrightarrow P^*(X)$ where for every $n \in X$, $T(1) = \{1\}, T(2) = \{1, 2\}, T(3) = \{3\}, T(4) = \{4\}, T(5) = T(6) = \{1, 5, 6\}$.

(1)Let $A = \{1, 3, 5\}$ then $T^+(A) = \{1, 3\}$, $\underline{P}(A) = \frac{2}{6}$ and $T^{-1}(A) = \{1, 2, 3, 5, 6\}$, $P(A) = \frac{3}{6}$ and $\overline{P}(A) = \frac{5}{6}$.

Tabel 2.1: Lower and upper probabilities of a random variable U

u	1	2	3	4	5	6
$\underline{P}(U = u)$	$\frac{1}{6}$	0	$\frac{1}{6}$	$\frac{1}{6}$	0	0
$\overline{P}(U = u)$	$\frac{4}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{2}{6}$

Definition 2.5 Let $T : X \longrightarrow P^*(X)$ be a set-valued mapping and A, B be two events in the stochastic approximation space $S = (X, P)$. We define $\underline{P}(A|B) = \frac{\underline{P}(A \cap B)}{\underline{P}(B)}$ for every $\underline{P}(B) \neq 0$ and $\overline{P}(A|B) = \frac{\overline{P}(A \cap B)}{\overline{P}(B)}$ for every $\overline{P}(B) \neq 0$.

Lemma 2.6 Let $T : X \longrightarrow P^*(X)$ be a set-valued mapping and A, B, C be three events in the stochastic approximation space $S = (X, P)$. then the following holds:

(1)$\underline{P}(A|A) = \overline{P}(A|A) = 1$;
(2)$\underline{P}(\emptyset|A) = \overline{P}(\emptyset|A) = 0$;
(3)$\underline{P}(A|X) = \underline{P}(A)$ and $\overline{P}(A|X) = \overline{P}(A)$;
(4)$\underline{P}(A^c|B) \leq 1 - \underline{P}(A|B)$;
(5)$\underline{P}(A \cup B|C) \geq \underline{P}(A|C) + \underline{P}(B|C) - \underline{P}(A \cap B|C)$;
(6)$\overline{P}(A^c|B) \geq 1 - \overline{P}(A|B)$;
(7)$\overline{P}(A \cup B|C) \leq \overline{P}(A|C) + \overline{P}(B|C) - \overline{P}(A \cap B|C)$;
(8)$\underline{P}(A) \geq \sum_{i=1}^{n} \underline{P}(A|B_i)\underline{P}(B_i)$, where $\bigcup_{i=1}^{n} B_i = X$;
(9)$\overline{P}(A) \leq \sum_{i=1}^{n} \overline{P}(A|B_i)\overline{P}(B_i)$, where $\bigcup_{i=1}^{n} B_i = X$;
(10)If T has transitive property and if B is an exact subset of X then

$$\underline{P}(A|B) \leq P(A|B) \leq \overline{P}(A|B).$$

Example 2.7 Consider the same experiment as in Example 2. Let $X = \{1, 2, 3, 4, 5, 6\}$, $B = \{1, 3, 5\}$ and $A = \{4, 5, 6\}$ then

$$\underline{P}(A|B) = \frac{\underline{P}(\{5\})}{\underline{P}(\{1, 3, 5\})} = 0,$$

and

$$\overline{P}(A|B) = \frac{\overline{P}(\{5\})}{\overline{P}(\{1, 3, 5\})} = \frac{\frac{2}{6}}{\frac{5}{6}} = \frac{2}{5}.$$

We define the lower and upper distribution functions of a random variable U.

Definition 2.8 Let $T : X \longrightarrow P^*(X)$ be a set-valued mapping and U be a random variable in the stochastic approximation space $S = (X, P)$. The lower and upper distribution of U is given by:

$$\underline{F}(u) = \underline{P}(U \leq u) \quad ; \quad \overline{F}(u) = \overline{P}(U \leq u),$$

respectively.

Definition 2.9 Let $T : X \longrightarrow P^*(X)$ be a set-valued mapping and U be a random variable in the stochastic approximation space $S = (X, P)$. The rough distribution function of U, denoted by $F^*(u)$, is given by:

$$F^*(u) = (\underline{F}(u), \overline{F}(u)).$$

Definition 2.10 Consider the same experiment as in Example 2. The Lower and upper distribution function of U are

$$\underline{F}(u) = \begin{cases} 0, & -\infty < u < 1, \\ \frac{1}{6}, & 1 \leq u < 3, \\ \frac{2}{6}, & 3 \leq u < 4, \\ \frac{3}{6}, & 4 \leq u < \infty \end{cases}$$

And

$$\overline{F}(u) = \begin{cases} 0, & -\infty < u < 1, \\ \frac{4}{6}, & 1 \leq u < 2, \\ \frac{5}{6}, & 2 \leq u < 3, \\ 1, & 3 \leq u < 4, \\ \frac{8}{6}, & 4 \leq u < 5, \\ \frac{9}{6}, & 5 \leq u < 6, \\ \frac{11}{6}, & 6 \leq u < \infty. \end{cases}$$

Therefore $F^*(2) = (\frac{1}{6}, \frac{5}{6})$.

We define the lower and upper expectations of a random variable U in the stochastic approximation space $S = (X,P)$.

Definition 2.11 Let $T : X \longrightarrow P^*(X)$ be a set-valued mapping and U be a random variable in the stochastic approximation space $S = (X,P)$. The lower and upper expectation of U is given by:

$$\underline{E}(u) = \sum_{k=1}^{n} u_k \underline{P}(U = u_k) \quad ; \quad \overline{E}(u) = \sum_{k=1}^{n} u_k \overline{P}(U = u_k),$$

respectively.

Definition 2.12 Let $T : X \longrightarrow P^*(X)$ be a set-valued mapping and U be a random variable in the stochastic approximation space $S = (X,P)$. The rough expectation of U is denoted by $E^*(U)$ and is given by:

$$E^*(U) = (\underline{E}(U), \overline{E}(U)).$$

Example 2.13 Consider the same experiment as in Example 2. Then the lower and upper expectations of U are

$$\underline{E}(U) = 1 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} = \frac{4}{3},$$

and

$$\overline{E}(U) = 1 \cdot \frac{4}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{2}{6} + 6 \cdot \frac{2}{6} = \frac{35}{6}.$$

Hence rough expectation of U is

$$E^*(U) = (\frac{4}{3}, \frac{35}{6}).$$

Theorem 2.14 Let $T : X \longrightarrow P^*(X)$ be a set-valued mapping and U be a random variable in the stochastic approximation space $S = (X,P)$. For any constants a and b, we have

$$\underline{E}(aU + b) = a\underline{E}(U) + bc \text{ where } 0 \leq c \leq 1.$$

Proof.

$$\begin{aligned} \underline{E}(aU + b) &= \sum_{k=1}^{n} (au_k + b)\underline{P}(u_k) = \sum_{k=1}^{n} (au_k\underline{P}(u_k) + b\underline{P}(u_k) \\ &= a \sum_{k=1}^{n} u_k\underline{P}(u_k) + b \sum_{k=1}^{n} \underline{P}(u_k) \\ &= a\underline{E}(U) + bc \;\; where \;\; c = \sum_{k=1}^{n} \underline{P}(u_k)(i.e \;\; 0 \leq c \leq 1). \end{aligned}$$

Theorem 2.15 Let $T : X \longrightarrow P^*(X)$ be a set-valued mapping and U be a random variable in the stochastic approximation space $S = (X,P)$. For any constants a and b, we have

$$\overline{E}(aU + b) = a\overline{E}(U) + bd \text{ where } 1 \leq d \leq n, \quad n \in \mathbb{N}.$$

*Proof.*The proof is similar to Theorem 2.

We define the lower and upper variances of a random variable U in the stochastic approximation space $S = (X,P)$.

Definition 2.16 Let $T : X \longrightarrow P^*(X)$ be a set-valued mapping and U be a random variable in the stochastic approximation space $S = (X,P)$. The lower and upper variance of U is given by:

$$\underline{V}(U) = \underline{E}(U - \underline{E}(U))^2 \quad ; \quad \overline{V}(U) = \overline{E}(U - \overline{E}(U))^2,$$

respectively.

Definition 2.17 Let $T : X \longrightarrow P^*(X)$ be a set-valued mapping and U be a random variable in the stochastic approximation space $S = (X,P)$. The rough variance of U is denoted by $V^*(U)$ and is given by:

$$V^*(U) = (\underline{V}(U), \overline{V}(U)).$$

Example 2.18 Consider the same experiment as in Example 2. Then the lower and upper variances of U are

$$\underline{V}(U) = 0.4, \;\; \overline{V}(U) = 13.75.$$

The rough variance of U is $V^*(U) = (0.4, 13.75)$.

Theorem 2.19 Let $T : X \longrightarrow P^*(X)$ be a set-valued mapping and U be a random variable in the stochastic approximation space $S = (X,P)$. Then

$$\underline{V}(U) = \underline{E}(U)^2 - (2 - c)(\underline{E}(U))^2 \quad where \quad c = \sum_{k=1}^{n} \underline{P}(u_k).$$

Proof. We have

$$\begin{aligned} \underline{E}(U - \underline{E}(U))^2 &= \underline{E}(U^2 - 2U\underline{E}(U) + (\underline{E}(U))^2) \\ &= \underline{E}(U)^2 - 2\underline{E}(U)\underline{E}(U) + c(\underline{E}(U))^2 \quad where \quad c = \sum_{k=1}^{n} \underline{P}(u_k) \\ &= \underline{E}(U)^2 - 2(\underline{E}(U))^2 + c(\underline{E}(U))^2 \\ &= \underline{E}(U)^2 - (2 - c)\underline{E}(U)^2. \end{aligned}$$

Theorem 2.20 Let $T : X \longrightarrow P^*(X)$ be a set-valued mapping and U be a random variable in the stochastic approximation space $S = (X,P)$. Then

$$\overline{V}(U) = \overline{E}(U)^2 - (2 - d)(\overline{E}(U))^2 \quad where \quad d = \sum_{k=1}^{n} \overline{P}(u_k).$$

Proof. The proof is similar to Theorem 2.

Theorem 2.21 Let $T : X \longrightarrow P^*(X)$ be a set-valued mapping and U be a random variable in the stochastic approximation space $S = (X, P)$. For any constants a and b, we have

$$\underline{V}(aU+b) = a^2\underline{E}(U^2) - (2a-c)(\underline{E}(U))^2 + 2b(a-c)\underline{E}(U) + b^2c \quad \textit{where} \quad c = \sum_{k=1}^{n} \underline{P}(u_k).$$

Proof. We have

$$\begin{aligned}
\underline{V}(aU+b) &= \underline{E}((aU+b) - \underline{E}(U))^2 \\
&= \underline{E}(aU+b)^2 - 2(aU+b)\underline{E}(U) + (\underline{E}(U))^2 \\
&= \underline{E}(a^2U^2 + 2abU + b^2 - 2aU\underline{E}(U) - 2b\underline{E}(U) + (\underline{E}(U))^2) \\
&= a^2\underline{E}(U^2) + 2ab\underline{E}(U) + b^2c - 2a(\underline{E}(U))^2 - 2bc\underline{E}(U) + c(\underline{E}(U))^2 \\
&= a^2\underline{E}(U^2) - (2a-c)(\underline{E}(U))^2 + 2b(a-c)\underline{E}(U) + b^2c,
\end{aligned}$$

$\textit{where} \quad c = \sum_{k=1}^{n} \underline{P}(u_k).$

Theorem 2.22 Let $T : X \longrightarrow P^*(X)$ be a set-valued mapping and U be a random variable in the stochastic approximation space $S = (X, P)$. For any constants a and b, we have

$$\overline{V}(aU+b) = a^2\overline{E}(U^2) - (2a-d)(\overline{E}(U))^2 + 2b(a-d)\overline{E}(U) + b^2d \quad \textit{where} \quad d = \sum_{k=1}^{n} \overline{P}(u_k).$$

Proof. We have

$$\begin{aligned}
\overline{V}(aU+b) &= \overline{E}((aU+b) - \overline{E}(U))^2 \\
&= \overline{E}(aU+b)^2 - 2(aU+b)\overline{E}(U) + (\overline{E}(U))^2 \\
&= \overline{E}(a^2U^2 + 2abU + b^2 - 2aU\overline{E}(U) - 2b\overline{E}(U) + (\overline{E}(U))^2) \\
&= a^2\overline{E}(U^2) + 2ab\overline{E}(U) + b^2c - 2a(\overline{E}(U))^2 - 2bc\overline{E}(U) + c(\overline{E}(U))^2 \\
&= a^2\overline{E}(U^2) - (2a-d)(\overline{E}(U))^2 + 2b(a-d)\overline{E}(U) + b^2d, \quad \textit{where} \quad d = \sum_{k=1}^{n} \overline{P}(u_k)
\end{aligned}$$

3 Conclusion

Luay concerned the rough probability by using the notion of Topology. But in this paper, we considered and studied the rough probability in the stochastic approximation spaces by using set-valued mapping and obtain results on rough expectation, and rough variance. There have been many researches on the connection between rough sets and algebraic systems. But we studied stochastic approximation spaces from topological view that generalize the stochastic approximation space in the case of general relation.

We considered the relation between the set-valued mapping and U, which is the random variable in the stochastic approximation space $S = (X, P)$, and interpreted the lower and upper approximations of a subset of $S = (X, P)$ related to them.

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