Journal of Statistics Applications & Probability
An International Journal

http://dx.doi.org/10.18576/jsap/050108

Reliability Estimation for Negative Binomial Distribution Under Type-II Censoring Scheme

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Received: 16 Aug. 2015, Revised: 29 Oct. 2015, Accepted: 2 Nov. 2015
Published online: 1 Mar. 2016

Abstract: The negative binomial distribution is a well recognized lifetime model. In this paper, we consider estimation of reliability measures of this model using type-II censored data. We obtain maximum likelihood and Bayes estimates of parameter, reliability function and hazard rate of this model. We also provide asymptotic, bootstrap and Bayesian credible intervals for the parameter. Finally, we give numerical illustration based on simulation study.

Keywords: Nagetive Bimomial distribution; Type-II censoring; Reliability; Hazard rate; Maximum Likelihood estimator; Bayes Estimator.

1 Introduction

In reliability analysis, often the failures are noted at regular intervals. In such situations, it is recommended to apply any discrete distribution for failure times. The negative binomial distribution (NBD) is a well recognized lifetime model, advocated by many authors. The probability mass function (*pmf*) of a random variable X, following NBD with parameters $(r,\ \theta)$ Johnson et. al. [1], is given by

$$p\,(X=x)=\binom{x+r-1}{x}\,\theta^{r}\,(1-\theta)^{x}\,;\quad x>0\,,r>0\,,\ 0<\theta<1. \tag{1}$$

The reliability function of NBD with *pmf* (1), at a mission time t, is given by

$$R(t)=\sum_{y=t+1}^{\infty}\binom{y+r-1}{y}\,\theta^{r}\,(1-\theta)^{y}\,. \tag{2}$$

According to Barlow [2] hazard rate $r(t)$ of NBD, at time t (>0), comes out to be

$$\begin{aligned} h(t) &= 1-\frac{R(t)}{R(t-1)}, \\ &= 1-\frac{\sum_{y=t+1}^{\infty}\binom{y+r-1}{y}\,\theta^{r}\,(1-\theta)^{y}}{\sum_{y=t}^{\infty}\binom{y+r-1}{y}\,\theta^{r}\,(1-\theta)^{y}}. \end{aligned}$$

The negative binomial distribution has a long history of applications. The fitting of this distribution to various types of data is considered by Fisher [3], Bliss and Fisher [4] and Simon [5]. Schader and Schmid [6] obtained the ML estimator of θ using grouped data. They also compared the convergence of Newton Raphson versus EM algorithm. Adamidis [7]

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also obtained the estimate of parameter of this model using EM algorithm. Duvall [8] derived the Bayes estimator of parameter θ assuming the Beta model as prior distribution. Bradlow [9] derived the Bayes estimator of the parameter of NBD using conjugate prior and expressed the Bayes estimator in term of the ratio of two gamma functions using polynomial expression. Chaturvedi and Tomer [10] discussed the uniformly minimum variance unbiased estimation as well as Bayesian estimation of $P(X>Y)$, when both X and Y follow NBDs. Ganji [11] considered the Bayes estimation of parameters of generalized NBD and discussed its applications. Lio [12] presented the different forms of NBD and obtained Bayes estimates of parameter. Doherty [13] considered the parameter estimation for the interval censored data from NBD. Zhao [14] discussed the hypothesis testing of parameters of two negative binomial populations in case of missing data. Ganji [15] obtained the Bayes and empirical Bayes estimates of number of success using the left truncated Poisson distribution as prior of the same.
Censoring is defined as the loss of observations on the lifetime variable of interest in the process of an investigation. In life testing experiments, it may occur due to lack of time, scarcity of funds or any other unavoidable reason. In type-II censoring scheme some units (say n) are placed on test and the test is terminated after observing the lifetime of prefixed number, say $m(\leq n)$, of units. Thus out of n, the lifetimes of m units are observed and n-m are considered as censored [See Sinha [16], Lawless [17]) for details].
In this paper, we discuss estimation procedures for NBD with failure censored data. We provide ML and Bayes estimates of lifetime parameters, hazard rate and reliability functions. Rest of the paper is organized as follows. In Section 2, we present the likelihood function and derive MLEs of parameter, hazard rate and reliability function. In section 3, we evaluate the asymptotic and boot-p confidence intervals for the parameter. In Section 4, we give procedures to obtain Bayes estimates of these parametric functions using Metropolis-Hasting algorithm. We carry out simulation study in Section 5 and finally conclude the findings in Section 6.

2 Maximum Likelihood Estimation

Let n units are put to test and the lifetimes of each unit follows NBD with *pmf* (1). The test is terminated after observing the lifetimes of $m(\leq n)$ units. Suppose that the sample $\underline{x} = x_1 \leq x_2 \leq ... \leq x_m$ is observed after the termination of the experiment which has d 'tie runs' with length z_h for h^{th} one, $h = 1, 2, ..., d$ with $\sum_{h=1}^{d} z_h = m$. Here the term 'tie runs' indicate a 'sub chain consisting of equal integers. Using the result of Gan and Bain [18], we write the likelihood function of θ, given $\underline{x}$ and r as follows.

$$L(\theta|\underline{x}, r) = \begin{cases} \sum\limits_{s=0}^{n-m} \frac{n!}{(n-m-s)!(z_d+s)!\prod\limits_{j=1}^{d-1} z_j!} \prod\limits_{i=1}^{m} p(x_i)\,[p(x_m)]^s\,[R(x_m)]^{n-m-s} & ; \quad 0<\theta<1 \\ \text{if } \mathrm{t_1 < t_2 < .... < t_k \text{ has d tie} - \text{runs with length } z_j \text{ for the } j^{th} \text{ one}, \; j = 1, 2, ..., r,} & \\ \qquad 0 \qquad otherwise & \end{cases} \tag{3}$$

Using equation (1) and (2) the likelihood function (3) can be re-expressed as

$$L(\theta|\underline{x}, r) = \sum_{s=0}^{n-m} \frac{n!}{(n-m-s)!(z_d+s)!\prod\limits_{j=1}^{d-1}(z_j)!} \prod_{i=1}^{m} \binom{x_i+r-1}{x_i} \theta^r (1-\theta)^{x_i} \left\{ \binom{x_m+r-1}{x_m} \theta^r (1-\theta)^{x_m} \right\}^s$$
$$\left\{ \sum_{y=x_m+1}^{\infty} \binom{y+r-1}{y} \theta^r (1-\theta)^y \right\}^{n-m-s} \tag{4}$$

In order to obtain MLE of θ,we solve the likelihood equation

$$\frac{\partial}{\partial\theta} \log(L(\theta|\underline{x})) = 0,$$

that is,

$$\frac{rm}{\hat{\theta}} - \frac{\sum\limits_{i=1}^{m} x_i}{(1-\hat{\theta})} + \frac{\sum\limits_{s=0}^{n-m} C_s \xi^s(x_m,\hat{\theta})\phi^{n-m-s}(x_m,\hat{\theta}) \left(\frac{rs}{\hat{\theta}} - \frac{x_m s}{1-\hat{\theta}} + \frac{(n-m-s)\phi'(x_m,\hat{\theta})}{\phi(x_m,\hat{\theta})} \right)}{\sum\limits_{s=0}^{n-m} C_s \xi^s(x_m,\hat{\theta})\phi^{n-m-s}(x_m,\hat{\theta})} = 0, \tag{5}$$

where $\hat{\theta}$ is MLE of θ, $C_s = \frac{n!}{(n-m-s)!(z_d+s)!\prod_{j=1}^{d-1}(z_j)!}$, $\phi(x_m, \hat{\theta}) = \sum_{y=x_m+1}^{\infty} \binom{y+r-1}{y} \hat{\theta}^r (1-\hat{\theta})^y$ and

$\phi'(x_m, \hat{\theta}) = \sum_{y=x_m+1}^{\infty} \binom{y+r-1}{y} \hat{\theta}^{r-1} (1-\hat{\theta})^{y-1} \left(r-(y-r)\hat{\theta}\right)$ and $\xi(x_m, \hat{\theta}) = \binom{x_m+r-1}{x_m} \hat{\theta}^r (1-\hat{\theta})^{x_m}$.

We observe that (5) can not be solved analytically. Therefore, we use iteration method to obtain $\hat{\theta}$.
Remark: The ML estimates of the reliability function and hazard rate, at a given time t are given, respectively, by

$$\hat{R}(t) = \sum_{y=t+1}^{\infty} \binom{y+r-1}{y} \hat{\theta}^r (1-\hat{\theta})^y$$

and

$$\hat{h}(t) = 1 - \frac{\hat{R}(t)}{\hat{R}(t-1)}.$$

3 Confidence Interval:

The exact distribution of MLE of θ cannot be obtained explicitly. Therefore, we evaluate asymptotic confidence interval and bootstrap confidence interval for θ as follows:

3.1 Asymptotic Confidence Interval

Using, the asymptotic normality of MLE, we construct the asymptotic confidence intervals (ACI) for the parameterθ.The confidence limits for $100(1-\alpha)\%$ ACI are given by

$$\hat{\theta} \pm Z_{\alpha/2}\sqrt{\hat{I}^{-1}(\hat{\theta})},$$

where, $Z_{\alpha/2}$ is upper $100(\alpha/2)$ percentile of standard normal distribution and $I(\hat{\theta})$ is the observed Fisher's information given by

$$\begin{aligned}\hat{I}(\theta) &= \left.\frac{\partial^2 \log(L(\theta|\underline{x}, r))}{\partial\theta^2}\right|_{\theta=\hat{\theta}}. \\ &= -\frac{rm}{(1-\theta)^2} - \frac{\sum_{i=1}^{m} x_i}{\theta^2} - \left\{\frac{\sum_{s=0}^{n-m} C_s \xi^s \phi^{n-m-s} \frac{A(\theta,x_m)}{\theta(1-\theta)}}{\sum_{s=0}^{n-m} C_s \xi^s \phi^{n-m-s}}\right\}^2 - \frac{\sum_{s=0}^{n-m} \frac{C_s}{\theta(1-\theta)} \xi^s \phi^{n-m-s}}{\sum_{s=0}^{n-m} C_s \xi^s \phi^{n-m-s}} \\ &\quad \times \left.\left[A(\theta,x_m)\left\{\frac{rs-1}{\theta} - \frac{x_m s-1}{1-\theta} - (n-m-s)\frac{\phi'}{\phi}\right\} + A'_\theta(\theta,x_m)\right]\right|_{\theta=\hat{\theta}}\end{aligned}$$

where,

$$\phi = \phi(x_m, \theta),\ \phi' = \frac{\partial}{\partial\theta}\phi,\ \phi'' = \frac{\partial^2}{\partial\theta^2}\phi,\ \xi = \xi(x_m, \theta), A(\theta, x_m) = x_m s(1-\theta) - rs\theta - (n-m-s)\theta(1-\theta)(1-\phi)^{-1}\phi'$$

and

$$A'_\theta(\theta, x_m) = -x_m s - rs - (n-m-s)(1-\phi)^{-1}\phi'\left\{1-\theta(1-\theta)\phi'(1-\phi)\right\} - (n-m-s)\theta(1-\theta)(1-\phi)^{-1}\phi''.$$

3.2 Bootstrap confidence interval

Here, we evaluate the parametric bootstrap confidence intervals proposed by Efron and Tibshirani [19]. The bootstrap method is very useful when an assumption regarding the normality is not valid. In order to obtain boot-p confidence intervals, the computational algorithm is given as follows.

1. Compute the MLE $\hat{\theta}$ using the given type II censored sample.
2. Using $\hat{\theta}$ generate failure censored sample $\{x_1^*, x_2^*, ..., x_m^*\}$ of size m from $f(x;\hat{\theta})$.
3. Using sample obtained in step (2), compute the bootstrap estimate of θ, say $\hat{\theta}^*$.
4. Repeat step 2-3, B times, to get the set of bootstrap estimators $(\hat{\theta}_j^*;\, j=1,2,...B)$.
5. Arrange $(\hat{\theta}_j^*;\, j=1,2,...B)$ in ascending order and get $\left(\hat{\theta}_{[1]}^*, \hat{\theta}_{[2]}^*, \ldots, \hat{\theta}_{[B]}^*\right)$.
6. A two-sided $100\,(1-\alpha)\%$ boot-p confidence interval is given by,

   $$\left(\hat{\theta}_L^*, \hat{\theta}_U^*\right) = \left(\hat{\theta}_{[B(\alpha/2)]}^*, \hat{\theta}_{[B(1-\alpha/2)]}^*\right).$$

   where, $[q]$ denote the integer part of q.

4 Bayesian Estimation

Here, we assume that θ is a continuous random variable and follows Bata distribution with parameters (α,β), given by

$$\pi(\theta) = \frac{1}{B(\alpha,\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}; \qquad 0<\theta<1,\ \alpha,\beta>0. \tag{6}$$

Using the likelihood function and the prior distribution, given in (4) and (6) respectively, we obtain the following posterior distribution of θ.

$$\Pi\left(\theta\,|\underline{x}\right) \propto \sum_{s=0}^{n-m} \frac{n!}{(n-m-s)!(z_d+s)!\prod_{j=1}^{d-1}(z_j)!}\left\{\binom{x_m+r-1}{x_m}\right\}^s \left(\sum_{y=x_m+1}^{\infty}\binom{y+r-1}{y}\theta^r(1-\theta)^y\right)^{n-m-s}$$
$$\theta^{r(m+s)+\alpha-1}(1-\theta)^{\sum\limits_{i=1}^{m}x+x_m s+\beta-1}, \tag{7}$$

We observe that the posterior distribution of θ in (7) do not appear in closed form. We therefore use Markov Chain Moto Carlo method to evaluate Bayes estimate of θ. We apply Metropolis-Hastings (M-H) algorithm to generate sample observations from the posterior distribution of θgiven in (7). The M-H algorithm generates a sequence of observations from any distribution $f(\theta)$ as follows [See, Metropolis and Ulam [20]].

1. Start with any initial value $\theta_o \in (0,\,1)$.
2. Generate a candidate point θ^* form $q \sim N(\hat{\theta}, I(\hat{\theta}))$and a point u from *U(0, 1)*.
3. Calculate
   $$\alpha = \min\left(\frac{f(\theta^*)\,q(\theta_{t-1})}{f(\theta_{t-1})\,q(\theta^*)},1\right)$$
4. Set $\theta_t = \theta^*$with probabilityα else $\theta_t = \theta_{t-1}$.
5. Repeating the steps 2-4 for h' times, where h' is a large number, we get the sample observations $\theta^{(1)}, \theta^{(2)}, ..., \theta^{(h')}$.

After the burn-in process, we obtain samples of h observations from θ. Using these observations we can obtain the Bayes estimates of these parameters, as well as, we can evaluate the credible intervals.

4.1 Bayesian Intervals

Once the sample observations are generated from the posterior of θ, it can be used to obtain the credible intervals for the parameters. In this section, we provide procedure, based on the algorithm of Chen and Shao [21], to evaluate credible intervals and highest posterior density (HPD) intervals.
The essential steps of the algorithm are as follows.

(a) *Credible Intervals*

(i) Order the sample observations generated through M-H algorithm,

$$\theta_{(1)} \leq \theta_{(2)} \leq ...... \leq \theta_{(h)}$$

(ii) The $100\,(1-\alpha)\,\%$ Bayesian credible interval for θ is given, by

$$\left(\theta_{[(\alpha/2)h]}, \theta_{[(1-\alpha/2)h]}\right)$$

(b) *Highest posterior density (HPD) intervals*

(i) Find all possible $100\,(1-\alpha)\,\%$ credible intervals with their respective lengths as follows

$$\left(\theta_{(j)},\ \theta_{(j+[(1-\alpha)h])}\right), \qquad l_j = \theta_{(j+[(1-\alpha)h])} - \theta_{(j)}; \qquad j = 1, 2, ..., \alpha h$$

. (ii) Search for the credible intervals having smallest length for l_j^{θ}. This smallest length credible interval is HPD interval for θ.

5 Simulation Study

Here we present some numerical illustrations based on simulation study. We have chosen the values of θ and r to be $(0.3,\ 5)$, $(0.5,\ 5)$, $(0.3,\ 8)$ and $(0.5,\ 8)$. We generate 3000 samples using these parametric values and evaluate average MLE of θ along with its MSE. We also evaluate ACI and boot-p confidence intervals along with their coverage probabilities (*CP*).
In Bayesian study, we set the hyperparameters of Beta distribution to be $\alpha = 2$ and $\beta = 3$. For these values a plot of prior and posterior distributions is given in Figure 1. We obtain the Bayes estimates of parameter and reliability function along with their MSEs. The estimates of parameters, reliabilities and hazard rate are mentioned in Table 1-4, for different values of parameters. The average length of ACI, Bayesian credible and HPD intervals with their CPs are given in Table 5 and 6.
In this Simulation study, we conclude that

1. The MSE of MLE and Bayes estimates decreases as *m* increases for a given value of *n*.
2. While comparing boot-p and ACI, for the given value of *(n, m)*, we observe that boot-p CI provide better CP then ACI, although it has larger length.
3. The HPD interval has less values of length than credible intervals. The coverage probability in case of HPD is greater than credible intervals.

We also consider the analysis of a simulated data set to show how one can use the results, obtained in the previous sections, to a real life problem. We generate a sample of size 20 from the NBD with probability of success $\theta = 0.35$ and $r = 6$. The first 15 observations are 2, 3, 4, 4, 4, 5, 5, 5, 5, 6, 6, 7, 8, 8 and 9. The reliability function and hazard rate are evaluated at the given time $t_0 = 8$. On the basis of the this sample, the various estimates are given Table 7.

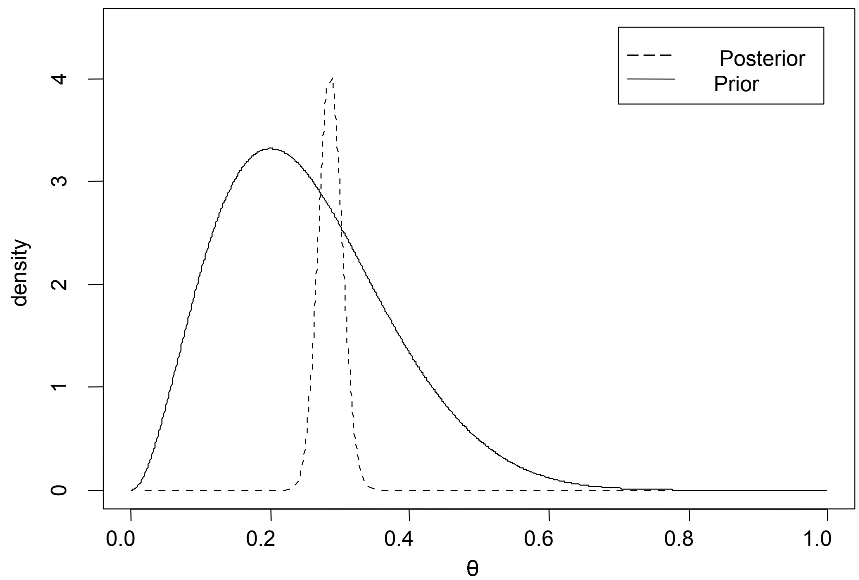

Figure 1 : Prior and posterior densities of θ.

Table 1: Average values of ML and Bayes estimator of θ, $R(t)$ and $h(t)$ with their MSE $\times 10^{-2}$ (in Bracket)) for $\theta = 0.30, r = 5$ and $t = 10$.

n=30	$\hat{\theta}$	$\hat{R}(t)$	$\hat{h}(t)$	$\tilde{\theta}$	$\tilde{R}(t)$	$\tilde{h}(t)$
10	0.308 (0.092)	0.516 (1.007)	0.124 (0.084)	0.306 (0.050)	0.515 (0.546)	0.12 (0.045)
15	0.297 (0.056)	0.517 (0.640)	0.118 (0.049)	0.298 (0.038)	0.516 (0.423)	0.119 (0.033)
20	0.298 (0.044)	0.519 (0.506)	0.117 (0.038)	0.298 (0.032)	0.517 (0.364)	0.118 (0.028)
25	0.304 (0.043)	0.503 (0.487)	0.123 (0.038)	0.304 (0.030)	0.504 (0.336)	0.122 (0.027)
30	0.299 (0.041)	0.517 (0.476)	0.118 (0.036)	0.300 (0.026)	0.516 (0.293)	0.119 (0.023)
n=50						
10	0.308 (0.076)	0.49 (0.844)	0.126 (0.072)	0.306 (0.043)	0.497 (0.462)	0.124 (0.04)
20	0.302 (0.044)	0.509 (0.504)	0.121 (0.040)	0.302 (0.031)	0.510 (0.351)	0.120 (0.029)
30	0.301 (0.040)	0.513 (0.457)	0.120 (0.036)	0.298 (0.031)	0.512 (0.345)	0.119 (0.027)
40	0.303 (0.027)	0.515 (0.303)	0.118 (0.024)	0.301 (0.025)	0.514 (0.237)	0.119 (0.02)
50	0.302 (0.026)	0.509 (0.300)	0.119 (0.023)	0.301 (0.021)	0.508 (0.241)	0.120 (0.019)

Table 2: Average values of ML and Bayes estimator of θ, $R(t)$ and $h(t)$ with their MSE $\times 10^{-2}$ (in Bracket) for $\theta = 0.50, r = 5$ and $t = 4$.

n=30 .	$\hat{\theta}$	$\hat{R}(t)$	$\hat{h}(t)$	$\tilde{\theta}$	$\tilde{R}(t)$	$\tilde{h}(t)$
10	0.503 (0.177)	0.626 (0.859)	0.183 (0.228)	0.497 (0.056)	0.636 (0.263)	0.178 (0.067)
15	0.502 (0.126)	0.630 (0.617)	0.181 (0.158)	0.498 (0.059)	0.636 (0.280)	0.178 (0.071)
20	0.505 (0.120)	0.623 (0.576)	0.184 (0.146)	0.502 (0.050)	0.631 (0.233)	0.180 (0.059)
25	0.502 (0.088)	0.631 (0.424)	0.180 (0.108)	0.499 (0.047)	0.635 (0.223)	0.178 (0.056)
30	0.498 (0.087)	0.634 (0.417)	0.179 (0.107)	0.501 (0.045)	0.637 (0.209)	0.177 (0.053)
n=50						
10	0.506 (0.160)	0.619 (0.775)	0.186 (0.204)	0.501 (0.054)	0.632 (0.253)	0.180 (0.064)
20	0.501 (0.077)	0.632 (0.364)	0.179 (0.091)	0.499 (0.040)	0.637 (0.187)	0.177 (0.047)
30	0.503 (0.066)	0.628 (0.324)	0.182 (0.082)	0.501 (0.038)	0.632 (0.186)	0.180 (0.047)
40	0.502 (0.055)	0.631 (0.264)	0.180 (0.066)	0.501 (0.038)	0.634 (0.175)	0.179 (0.043)
50	0.499 (0.055)	0.637 (0.257)	0.177 (0.063)	0.499 (0.036)	0.638 (0.169)	0.176 (0.042)

Table 3: Average values of ML and Bayes estimator of θ, $R(t)$ and $h(t)$ with their MSE $\times 10^{-2}$ (in Bracket) for $\theta = 0.30, r = 8$ and $t = 15$.

n=30	$\hat{\theta}$	$\hat{R}(t)$	$\hat{h}(t)$	$\tilde{\theta}$	$\tilde{R}(t)$	$\tilde{h}(t)$
10	0.303 (0.045)	0.604 (0.671)	0.083 (0.032)	0.303 (0.031)	0.604 (0.488)	0.083 (0.024)
15	0.302 (0.041)	0.614 (0.656)	0.081 (0.031)	0.301 (0.028)	0.613 (0.444)	0.081 (0.022)
20	0.301 (0.035)	0.613 (0.558)	0.081 (0.025)	0.302 (0.027)	0.611 (0.433)	0.081 (0.02)
25	0.302 (0.025)	0.609 (0.414)	0.081 (0.019)	0.302 (0.021)	0.607 (0.337)	0.082 (0.016)
30	0.301 (0.024)	0.614 (0.386)	0.080 (0.018)	0.301 (0.019)	0.612 (0.307)	0.081 (0.015)
n=50						
10	0.292 (0.05)	0.620 (0.789)	0.079 (0.037)	0.299 (0.036)	0.619 (0.561)	0.08 (0.027)
20	0.298 (0.035)	0.620 (0.562)	0.079 (0.026)	0.299 (0.028)	0.619 (0.449)	0.08 (0.021)
30	0.302 (0.027)	0.611 (0.438)	0.081 (0.021)	0.302 (0.022)	0.610 (0.365)	0.081 (0.017)
40	0.305 (0.021)	0.598 (0.350)	0.084 (0.016)	0.305 (0.019)	0.598 (0.306)	0.084 (0.015)
50	0.300 (0.015)	0.617 (0.242)	0.08 (0.011)	0.301 (0.013)	0.615 (0.211)	0.08 (0.010)

Table 4: Average values of ML and Bayes estimator of θ, $R(t)$ and $h(t)$ with their MSE $\times 10^{-2}$ (in Bracket) for $\theta = 0.50, r = 8$ and $t = 6$.

$n=30$	$\hat{\theta}$	$\hat{R}(t)$	$\hat{h}(t)$	$\tilde{\theta}$	$\tilde{R}(t)$	$\tilde{h}(t)$
10	0.497	0.611	0.146	0.496	0.614	0.145
	(0.078)	(0.646)	(0.096)	(0.041)	(0.325)	(0.048)
15	0.501	0.599	0.150	0.499	0.604	0.149
	(0.068)	(0.579)	(0.087)	(0.039)	(0.323)	(0.048)
20	0.498	0.610	0.146	0.497	0.611	0.146
	(0.061)	(0.502)	(0.074)	(0.040)	(0.326)	(0.048)
25	0.499	0.605	0.148	0.498	0.607	0.148
	(0.059)	(0.490)	(0.071)	(0.038)	(0.304)	(0.044)
30	0.500	0.603	0.149	0.499	0.605	0.148
	(0.037)	(0.318)	(0.047)	(0.025)	(0.209)	(0.031)
50						
10	0.508	0.580	0.158	0.503	0.592	0.154
	(0.092)	(0.798)	(0.125)	(0.050)	(0.407)	(0.061)
20	0.499	0.605	0.148	0.498	0.608	0.147
	(0.074)	(0.619)	(0.093)	(0.046)	(0.393)	(0.061)
30	0.506	0.587	0.155	0.504	0.591	0.153
	(0.040)	(0.351)	(0.053)	(0.028)	(0.245)	(0.037)
40	0.498	0.608	0.147	0.498	0.609	0.147
	(0.036)	(0.307)	(0.045)	(0.029)	(0.239)	(0.035)
50	0.499	0.607	0.147	0.501	0.608	0.150
	(0.033)	(0.285)	(0.042)	(0.025)	(0.213)	(0.032)

Table 5: Average Length of ACI, boot-p, Bayesian credible and HPD intervals with their CP for r=5.

		$\theta=0.3$				$\theta=0.5$			
$n=30$		ACI	Boot-p	BCI	HPD	ACI	Boot-p	BCI	HPD
10	Length	0.113	0.115	0.074	0.073	0.164	0.166	0.094	0.083
	CP	0.947	0.951	0.954	0.986	0.947	0.952	0.953	0.983
15	Length	0.099	0.108	0.069	0.069	0.140	0.143	0.088	0.080
	CP	0.950	0.952	0.955	0.989	0.945	0.962	0.955	0.988
20	Length	0.090	0.095	0.066	0.063	0.127	0.131	0.082	0.077
	CP	0.953	0.952	0.959	0.992	0.950	0.964	0.958	0.992
25	Length	0.084	0.088	0.063	0.061	0.119	0.123	0.078	0.075
	CP	0.957	0.953	0.958	0.991	0.959	0.963	0.962	0.995
30	Length	0.081	0.081	0.062	0.058	0.113	0.115	0.074	0.073
	CP	0.953	0.955	0.961	0.994	0.963	0.968	0.968	0.996
$n=50$									
	Length	0.096	0.106	0.072	0.071	0.158	0.167	0.083	0.082
10	CP	0.952	0.961	0.957	0.986	0.958	0.956	0.947	0.989
	Length	0.082	0.098	0.063	0.060	0.119	0.129	0.075	0.068
20	CP	0.958	0.963	0.958	0.989	0.962	0.963	0.952	0.991
	Length	0.072	0.085	0.058	0.054	0.102	0.113	0.071	0.070
30	CP	0.959	0.964	0.961	0.988	0.964	0.965	0.958	0.993
	Length	0.066	0.076	0.054	0.051	0.093	0.107	0.067	0.059
40	CP	0.962	0.965	0.963	0.992	0.975	0.969	0.958	0.992
	Length	0.060	0.072	0.052	0.048	0.088	0.098	0.065	0.055
50	CP	0.968	0.968	0.965	0.996	0.979	0.968	0.962	0.996

Table 6: Average Length of ACI, boot-p, Bayesian credible and HPD intervals with their CP for r=8.

		$\theta = 0.3$				$\theta = 0.5$			
$n = 30$		ACI	Boot-p	BCI	HPD	ACI	Boot-p	BCI	HPD
10	Length	0.084	0.097	0.063	0.063	0.123	0.131	0.076	0.076
	CP	0.946	0.956	0.951	0.981	0.952	0.962	0.975	0.992
15	Length	0.076	0.087	0.060	0.059	0.108	0.117	0.072	0.072
	CP	0.948	0.957	0.952	0.982	0.952	0.965	0.976	0.994
20	Length	0.070	0.082	0.057	0.056	0.099	0.096	0.069	0.069
	CP	0.948	0.960	0.962	0.986	0.960	0.970	0.982	0.994
25	Length	0.066	0.079	0.055	0.054	0.093	0.093	0.067	0.067
	CP	0.946	0.964	0.968	0.992	0.961	0.976	0.986	0.995
30	Length	0.064	0.640	0.053	0.053	0.089	0.089	0.066	0.066
	CP	0.950	0.965	0.975	0.995	0.959	0.985	0.989	0.996
$n = 50$									
10	Length	0.079	0.092	0.061	0.061	0.115	0.118	0.074	0.068
	CP	0.946	0.957	0.949	0.985	0.949	0.958	0.952	0.991
20	Length	0.063	0.086	0.053	0.052	0.090	0.093	0.066	0.063
	CP	0.952	0.956	0.954	0.987	0.950	0.962	0.954	0.993
30	Length	0.062	0.072	0.048	0.046	0.079	0.088	0.061	0.059
	CP	0.953	0.961	0.962	0.987	0.956	0.966	0.956	0.995
40	Length	0.052	0.067	0.046	0.045	0.073	0.079	0.058	0.054
	CP	0.958	0.965	0.965	0.990	0.960	0.965	0.955	0.998
50	Length	0.052	0.065	0.046	0.043	0.069	0.072	0.058	0.054
	CP	0.959	0.968	0.969	0.992	0.965	0.971	0.964	0.998

Table 7: ML and Bayes estimates of the θ, Reliability, hazard rate for the given data.

$\hat{\theta}$	$\hat{R}(t)$	$\hat{h}(t)$	$\tilde{\theta}$	$\tilde{R}(t)$	$\tilde{h}(t)$	ACI	boot-p	BCI	HPD
0.351	0.638	0.106	0.351	0.635	0.108	$(0.297, 0.405)$	$(0.301, 0.402)$	$(0.306, 0.398)$	$(0.306, 0.398)$

6 Conclusion

In this paper, we discussed estimation procedures for parameter and reliability function of Negative Binomial distribution. We have obtained ML and Bayes estimates of parameter, reliability and hazard rate functions for this distribution using failure censored data. We also considered interval estimation of parameter. Here, we have provided ACI and bootstrap CI as well as Bayesian credible and HPD intervals for the parameter. We have conducted extensive simulation study in order to assess the performance of the estimators and presented result through figure and tables. We have mentioned the findings based on simulation study in Section 5.

Acknowledgement

The authors would like to thank the editor and learned referees for constructive suggestions which helped in improvement of the manuscript a lot.

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