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Some Characterizations of Relatively Strictly Semi-Monotone (Semi-Monotone) Operators and Applications

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Abstract: In [M. A. Tawhid, On Characterization of $\mathbf{E}(\mathbf{E0})$-Properties in Nonsmooth Functions. Applied Mathematics and Computation. 175: 2: pp. 1609-1618, April 15, (2006)], Tawhid gave characterization of strictly semi-monotone (semi-monotone) properties in nonsmooth functions that are H-differentiable. He showed the usefulness of his results to nonlinear complementarity problems. A natural question is: Can we extend these characterizations in order to apply the results to nonsmooth generlaized complementarity problems? This paper give an affirmative answer. We introduce the concepts of relatively semi-monotone and relatively strictly semi-monotone in order to give characterizations of the relatively semi-monotone and relatively strictly semi-monotone properties. Also, our results give characterizations of relatively $\mathbf{P}(\mathbf{P_0})$- when the underlying functions are C^1-functions, semismooth-functions, and locally Lipschitzian functions. Moreover, we show useful applications of our results by giving illustrations to nonsmooth generalized complementarity problems that admit the H-differentiability.

Keywords: H-Differentiability, semismooth-functions, locally Lipschitzian, generalized Jacobian, relatively strictly semi-monotone, relatively semi-monotone, generalized complementarity problems.
AMS subject classification: 49J52, 90C33, 90C46

1 Introduction

In [6], the authors introduced the concepts of the H-differentiability and H-differential for a function $f : R^n \to R^n$. They showed that the Fréchet derivative of a Fréchet differentiable function, the Clarke generalized Jacobian of a locally Lipschitzian function [1], the Bouligand subdifferential of a semismooth function [11], [17], [19], and the C-differential of a C-differentiable function [18] are examples of H-differentials. It turns out (see ([5], [6], [22], [25], [27], [26], [28], [29], [30]) that these concepts give useful and unified treatments for many problems in optimization, complementarity problems, and variational inequalities when the underlying functions are not necessarily locally Lipschitzian nor semismooth. Any superset of an H-differential is an H-differential, H-differentiability implies continuity, and H-differentials enjoy simple sum, product and chain rules, see [22]. The H-differentiable function need not be locally Lipschitzian nor directionally differentiable [25].

Our work in this article is motivated from some recent results: The characterization of $\mathbf{P}(\mathbf{P_0})$- properties in nonsmooth functions [22], the characterization $\mathbf{E}(\mathbf{E_0})$-properties in nonsmooth functions [23], and some applications of H-differentiability to optimization, complementarity, and variational inequalities [5], [6], [22], [25], [27], [26], [28], [29], [30].

The goal of this paper is to give a characterization of relatively semi-monotone $(\mathbf{E_0})$– and relatively strictly semi-monotone $(\mathbf{E})$– property when the underlying functions are H-differentiable. We establish our results by introducing the concepts of relatively semi-monotone $(\mathbf{E_0})$– and relatively strictly semi-monotone $(\mathbf{E})$ which extend the concepts of semi-monotone $(\mathbf{E_0})$– and strictly

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semi-monotone (**E**). Therefore, our results extend/generalize the characterization of **E**(**E0**)-Properties in nonsmooth functions in [23].

Also, we show the usefulness of our results by giving some illustrations to nonsmooth generalized complementarity problems.

2 Preliminaries

We regard vectors in R^n as column vectors. For a matrix A, A_i denotes the ith row of A. For a differentiable function $f : R^n \to R^m$, $\nabla f(\overline{x})$ denotes the Jacobian matrix of f at $\overline{x}$. Vector inequalities are interpreted componentwise. For a set $K \subseteq R^n$, coK denotes the convex hull of K and $\overline{K}$ denotes the closure of K [20]. The p-norm of x is denoted $||x||_p$ and the Euclidean norm of x is denoted by $||x||$.

2.1 H-differentiability and H-differentials

From [6], we recall the following definition.

Definition 1. *Given a function $F : \Omega \subseteq R^n \to R^m$ where Ω is an open set in R^n and $x^* \in \Omega$, we say that a nonempty subset $T(x^*)$ (also denoted by $T_F(x^*)$) of $R^{m\times n}$ is an H-differential of F at x^* if for every sequence $\{x^k\} \subseteq \Omega$ converging to x^*, there exist a subsequence $\{x^{k_j}\}$ and a matrix $A \in T(x^*)$ such that*

$$F(x^{k_j}) - F(x^*) - A(x^{k_j} - x^*) = o(||x^k_j - x^*||). \tag{1}$$

We say that F is H-differentiable at x^* if F has an H-differential at x^*.

Remarks. In [27], it is shown that if a function $F : \Omega \subseteq R^n \to R^m$ is H-differentiable at a point $\overline{x}$, then there exist a constant $L > 0$ and a neighbourhood $B(\overline{x},\delta)$ of $\overline{x}$ with

$$||F(x) - F(\overline{x})|| \leq L||x - \overline{x}||, \quad \forall x \in B(\overline{x},\delta). \tag{2}$$

Conversely, $T(\overline{x}) := R^{m\times n}$ can be taken as an H-differential of F at $\overline{x}$ if condition (2) holds. Thus (2) gives an alternate description of H-differentiability.

Obviously, any function locally Lipschitzian at $\overline{x}$ will satisfy (2). For real valued functions, condition (2) is known as the ‘calmness’ of F at $\overline{x}$. This concept has been well studied in the literature of nonsmooth analysis (see [21], Chapter 8).

The authors in [6] showed the Fréchet derivative of a Fréchet differentiable function, the Clarke generalized Jacobian of a locally Lipschitzian function, the Bouligand subdifferential of a semismooth function, and the C-differential of a C-differentiable function are particular examples of H-differentials.

Example 1. (Fréchet differentiability)

Let $F : R^n \to R^m$ be Fréchet differentiable at $x^* \in R^n$ with Fréchet derivative matrix (= Jacobian matrix derivative) $\{\nabla F(x^*)\}$ such that

$$F(x) - F(x^*) - \nabla F(x^*)(x - x^*) = o(||x - x^*||).$$

Then F is H-differentiable with $\{\nabla F(x^*)\}$ as an H-differential.

Example 2. (Locally Lipschitzian function)

Let $F : \Omega \subseteq R^n \to R^m$ be locally Lipschitzian at each point of an open set Ω. For $x^* \in \Omega$, define the Bouligand subdifferential of F at x^* by

$$\partial_B F(x^*) = \{\lim \nabla F(x^k) : x^k \to x^*, x^k \in \Omega_F\}$$

where Ω_F is the set of all points in Ω where F is Fréchet differentiable. Then, the (Clarke) generalized Jacobian [1]

$$\partial F(x^*) = co\partial_B F(x^*)$$

is an H-differential of F at x^*.

Example 3. (Semismooth function)

Consider a locally Lipschitzian function $F : \Omega \subseteq R^n \to R^m$ that is semismooth at $x^* \in \Omega$ [11], [17], [19]. This means for any sequence $x^k \to x^*$, and for $V_k \in \partial F(x^k)$,

$$F(x^k) - F(x^*) - V_k(x^k - x^*) = o(||x^k - x^*||).$$

Then the Bouligand subdifferential

$$\partial_B F(x^*) = \{\lim \nabla F(x^k) : x^k \to x^*, x^k \in \Omega_F\}.$$

is an H-differential of F at x^*. In particular, this holds if F is piecewise smooth, i.e., there exist continuously differentiable functions $F_j : R^n \to R^m$ such that

$$F(x) \in \{F_1(x), F_2(x), \ldots, F_J(x)\} \quad \forall x \in R^n.$$

Example 4. (C-differentiability)

Let $F : R^n \to R^n$ be C-differentiable [18] in a neighborhood D of x^*. This means that there is a compact upper semicontinuous multivalued mapping $x \mapsto T(x)$ with $x \in D$ and $T(x) \subset R^{n\times n}$ satisfying the following condition at any $a \in D$: For any $V \in T(x)$,

$$F(x) - F(a) - V(x - a) = o(||x - a||).$$

Then, F is H-differentiable at x^* with $T(x^*)$ as an H-differential.

Remark. It is noted that an H-differentiable function need not be locally Lipschitzian nor directionally differentiable. The following simple example, is taken from [25], consider on R,

$$F(x) = x\sin(\frac{1}{x}) \text{ for } x \neq 0 \text{ and } F(0) = 0.$$

Then F is H-differentiable on R with

$$T(0) = [-1,1] \text{ and } T(c) = \{\sin(\frac{1}{c}) - \frac{1}{c}cos(\frac{1}{c})\} \text{ for } c \neq 0.$$

We note that F is not locally Lipschitzian around zero. We also see that F is neither Fréchet differentiable nor directionally differentiable.

3 The relatively $\mathbf{E}(\mathbf{E_0})$– properties in nonsmooth functions

Let us recall the definitions of semi-monotone ($\mathbf{E_0}$) and strictly semi-monotone ($\mathbf{E}$) functions (matrices), see e.g., [4], [9], and [12].

Definition 2. *For a function $f : R^n \to R^n$, we say that f is semi-monotone ($\mathbf{E_0}$) if for every $0 \neq x \geq 0$ there exists an index j such that $x_j > 0$ and $f_j(x) \geq 0$. It is strictly semi-monotone ($\mathbf{E}$) if for every $0 \neq x \geq 0$ there exists an index j such that $x_j > 0$ and $f_j(x) > 0$.*

A matrix $M \in R^{n\times n}$ is said to be a $\mathbf{E_0}(\mathbf{E})$-matrix if the function $f(x) = Mx$ is a $\mathbf{E_0}(\mathbf{E})$-function.

The following Proposition in [23] is an analog of proposition in [22] for strictly semi-monotone ($\mathbf{E}$)–matrices.

Proposition 1. *Let $f : \Omega \to R^n$ be continuous where Ω is open set in R^n and H-differentiable at each point $\bar{x} \in \Omega$ with an H-differential $T(\bar{x})$ consisting of strictly semi-monotone matrices. Then there exists vectors u and v arbitrarily close to zero such that*

(i) $u < 0$ and $f(\bar{x}+u) < f(\bar{x})$;
(i) $v > 0$ and $f(\bar{x}+v) > f(\bar{x})$.

Now we introduce the definitions of relatively semi-monotone ($\mathbf{E_0}$) and relatively strictly semi-monotone ($\mathbf{E}$) functions (matrices).

Definition 3. *For a function $f,g : R^n \to R^n$, we say that f and g are relatively semi-monotone if for every $0 \neq g(x) \geq 0$ there exists an index j such that $g_j(x) > 0$ and $f_j(x) \geq 0$. It is relativelystrictly semi-monotone if for every $0 \neq g(x) \geq 0$ therelativelyre exists an index j such that $g_j(x) > 0$ and $f_j(x) > 0$.*

The following Lemma is needed in the subsequent analysis. The proof is trivial so we omit it.

Lemma 1. *Suppose $f,g : R^n \to R^n$ and g is one-to-one and onto. Define $h : R^n \to R^n$ where $h := f \circ g^{-1}$. Then f and g are relatively $\mathbf{E_0}(\mathbf{E})$-functions if and only if h is $\mathbf{E_0}(\mathbf{E})$-function.*

A continuous mapping is called a homeomorphism if it is a one-to-one and onto mapping and if its inverse mapping is also continuous.
The proof of the following theorem based on Proposition 1, is similar to the proofs of Theorem 3.4 in [9] and Theorem 1 in [23].

Theorem 1. *Let $Q \subseteq R^n{}_+$ be a rectangular box of the form $Q = \{x|\ 0 \leq x \leq a\}$. Suppose that $f : R^n \to R^n$ and $g : R^n \to R^n$ are continuous and H-differentiable at each $\bar{x} \in Q$ with H-differentials, respectively, by $T_f(\bar{x})$ and $T_g(\bar{x})$. Assume g is a homeomorphism. Let $h : Q \to R^n$ be continuous with $h(0) = 0$ where $h := f \circ g^{-1}$ and H-differentiable at each point $\bar{x} \in \Omega$ with an H-differential $T_h(\bar{x})$ consisting of strictly semi-monotone ($\mathbf{E}$)– matrices. Then h is a strictly semi-monotone ($\mathbf{E}$)–function on Q.*

In view of Lemma 1 and Theorem 1, we have the following.

Corollary 1. *Let $Q \subseteq R^n{}_+$ be a rectangular box of the form $Q = \{x|\ 0 \leq x \leq a\}$. Suppose that $f : R^n \to R^n$ and $g : R^n \to R^n$ are continuous and H-differentiable at each $\bar{x} \in Q$ with H-differentials, respectively, by $T_f(\bar{x})$ and $T_g(\bar{x})$. Assume g is a homeomorphism. Let $h : Q \to R^n$ be continuous with $h(0) = 0$ where $h := f \circ g^{-1}$ and H-differentiable at each point $\bar{x} \in \Omega$ with an H-differential $T_h(\bar{x})$ consisting of strictly semi-monotone ($\mathbf{E}$)– matrices. Then f and g are relatively strictly semi-monotone ($\mathbf{E}$)-functions on Q, i.e., for every $x \in Q, 0 \neq g(x) \geq 0$, there exists an index j such that $g_j(x) > 0$ and $f_j(x) > 0$.*

Remark. Note that if $g(x) = x$ in Corollary 1, we get Theorem 1 in [23].
The following theorem characterizes the relatively semi-monotone ($\mathbf{E_0}$)–property via H-differentials.

Theorem 2. *Let $Q \subseteq R^n{}_+$ be a rectangular box of the form $Q = \{x|\ 0 \leq x \leq a\}$. Suppose that $f : R^n \to R^n$ and $g : R^n \to R^n$ are continuous and H-differentiable at each $\bar{x} \in Q$ with H-differentials, respectively, by $T_f(\bar{x})$ and $T_g(\bar{x})$. Assume g is a homeomorphism. Let $h : Q \to R^n$ be continuous with $h(0) = 0$ where $h := f \circ g^{-1}$ and H-differentiable at each point $\bar{x} \in \Omega$ with an H-differential $T_h(\bar{x})$ consisting of semi-monotone ($\mathbf{E_0}$)–matrices. Then h is a semi-monotone ($\mathbf{E_0}$)– function on Q.*

In view of Lemma 1 and Theorem 2, we have the following.

Corollary 2. *Let $Q \subseteq R^n{}_+$ be a rectangular box of the form $Q = \{x|\ 0 \leq x \leq a\}$. Suppose that $f : R^n \to R^n$ and $g : R^n \to R^n$ are continuous and H-differentiable at each $\bar{x} \in Q$ with H-differentials, respectively, by $T_f(\bar{x})$ and $T_g(\bar{x})$. Assume g is a homeomorphism. Let $h : Q \to R^n$ be continuous with $h(0) = 0$ where $h := f \circ g^{-1}$ and H-differentiable at each point $\bar{x} \in \Omega$ with an H-differential $T_h(\bar{x})$ consisting of semi-monotone ($\mathbf{E_0}$)–matrices. Then f and g are relatively strictly semi-monotone ($\mathbf{E_0}$)-functions on Q.*

Remark. Note that if $g(x) = x$ in Corollary 2, we get Theorem 2 in [23]. In view of Example 2, we get the following.

Corollary 3. *Let $Q \subseteq R^n{}_+$ be a rectangular box of the form $Q = \{x|\ 0 \leq x \leq a\}$. Suppose that $f : R^n \to R^n$ and $g : R^n \to R^n$ are continuous and locally Lipschitzian at each $\bar{x} \in Q$ with generalized Jacobians, respectively, by $\partial f(\bar{x})$ and $\partial g(\bar{x})$. Assume g is a homeomorphism and $\partial g(\bar{x})$ consists of nonsingular matrices. Let $h : Q \to R^n$ be* *continuous with $h(0)=0$ where $h:=f\circ g^{-1}$ and locally Lipschitzian at each point $\bar{x}\in\Omega$ with generalized Jacobian $\partial h(\bar{x})$ consisting of semi-monotone $(\mathbf{E_0})$– matrices. Then*

(i) h is a semi-monotone $(\mathbf{E_0})$– function on Q.
(ii) f and g are relatively strictly semi-monotone $(\mathbf{E_0})$-functions on Q

In View of Example **??**, we get the following.

Corollary 4. *Let $Q\subseteq R^n{}_+$ be a rectangular box of the form $Q=\{x|\ 0\le x\le a\}$. Suppose that $f:R^n\to R^n$ and $g:R^n\to R^n$ are continuous and semismooth on R^n (in particular, piecewise affine or piecewise smooth) at each $\bar{x}\in Q$ with the Bouligand subdifferentials , respectively, by $\partial_B f(\bar{x})$ and $\partial_B g(\bar{x})$. Assume g is a homeomorphism and $\partial_B g(\bar{x})$ consists of nonsingular matrices. Let $h:Q\to R^n$ be continuous with $h(0)=0$ where $h:=f\circ g^{-1}$ and semismooth (in particular, piecewise affine or piecewise smooth) at each point $\bar{x}\in Q$ with Bouligand subdifferential $\partial_B h(\bar{x})$ consisting of semi-monotone $(\mathbf{E_0})$– matrices. Then*

(i) h is a semi-monotone $(\mathbf{E_0})$– function on Q.
(ii) f and g are relatively strictly semi-monotone $(\mathbf{E_0})$-functions on Q.

If $g(x)=x$ in the above corollaries, we have the following corollary.

Corollary 5. *Under each of the following, $f:R^n\to R^n$ with $f(0)=0$, is a strictly semi-monotone (semi-monotone) function.*

(a) f is Fréchet differentiable on R^n and for every $x\in R^n$, the Jacobian matrix $\nabla f(x)$ is a strictly semi-monotone (semi-monotone) matrix.
(b) f is locally Lipschitzian on R^n and for every $x\in R^n$, the generalized Jacobian $\partial f(x)$ consists of strictly semi-monotone (semi-monotone) matrices.
(c) f is semismooth on R^n (in particular, piecewise affine or piecewise smooth) and for every $x\in R^n$, the Bouligand subdifferential $\partial_B f(x)$ consists of strictly semi-monotone (semi-monotone) matrices.

4 Some applications to generalized complementarity problems

Before we start this section, we need the following definition

Definition 4. *A function $\phi:R^2\to R$ is called a GCP function if $\phi(a,b)=0\Leftrightarrow ab=0, a\ge 0, b\ge 0$. For the problem GCP$(f,g)$, we define*

$$\Phi(x)=\left[\phi(f_1(x),g_1(x)),\dots,\phi(f_n(x),g_n(x))\right]^T \qquad (3)$$

and, we call $\Phi(x)$ a GCP function for GCP(f,g).

In this section, we give some illustrations to generalized complementarity problems to illustrate the usefulness of our results. Given any two functions $f,g:R^n\to R^n$, the generalized complementarity problem GCP(f,g) is the problem of finding $\bar{x}\in R^n$ such that

$$f(\bar{x})\ge 0,\ g(\bar{x})\ge 0 \ \text{ and } f(\bar{x})^T g(\bar{x})=0.$$

The GCP(f,g) can be regarded as a generalization of some complementarity problems. Also, GCP(f,g) is known as the quasi/implicit complementarity problem when $g(x)=x-W(x)$ with some $W:R^n\to R^n$, see, e.g., [7], [13], [16].

We consider a GCP function $\Phi:R^n\to R^n$ associated with GCP(f,g) and its merit function

$$\Psi(x):=\frac{1}{2}||\Phi(x)||^2, \qquad (4)$$

so that

$$\bar{x} \text{ solves GCP}(f,g)\ \Leftrightarrow \Phi(\bar{x})=0\Leftrightarrow\Psi(\bar{x})=0.$$

In order to show the usefulness of our results, we need to know the H-differential of some GCP functions.

The following illustration is Theorem 2 in [24].

Example 5. Consider the GCP function based on the NCP function in [8]

$$\Phi(x):=f(x)+g(x)-\sqrt{[f(x)-g(x)]^2+\lambda f(x)g(x)} \qquad (5)$$

where λ is a fixed parameter in $(0,4)$. We note that when $\lambda\to 0$, $\Phi(x)$ becomes

$$\begin{cases}\Phi(x):=f(x)+g(x)-\sqrt{[f(x)-g(x)]^2}\\ \qquad =2\min\{f(x),g(x)\},\end{cases}$$

while $\lambda=2$, $\Phi(x)$ reduces to the GCP function based on Fischer-Burmeister function. Let

$$J(\bar{x})=\{i:\ f_i(\bar{x})=0=g_i(\bar{x})\}.$$

Then Φ in (5) has an H-differential at $\bar{x}$ given by

$$T_\Phi(\bar{x})=\{VA+WB:\ (A,B,V,W,d)\in\Gamma\},$$

where Γ is the set of all quintuples (A,B,V,W,d) with $A\in T_f(\bar{x})$, $B\in T_g(\bar{x})$, $||d||=1$, $V=diag(v_i)$ and $W=diag(w_i)$ are diagonal matrices satisfying the conditions

$$(1-v_i)^2+(1-w_i)^2\in(0,2)\ \ \forall i=1,2\dots,n, \quad \text{where} \qquad (6)$$

$$v_i=\begin{cases}1-\frac{-2(g_i(\bar{x})-f_i(\bar{x}))+\lambda g_i(\bar{x})}{2\sqrt{(g_i(\bar{x})-f_i(\bar{x}))^2+\lambda f_i(\bar{x})g_i(\bar{x})}} & \text{when } i\notin J(\bar{x})\\ 1-\frac{-2(B_id-A_id)+\lambda(B_id)}{2\sqrt{(A_id-B_id)^2+\lambda(A_id)(B_id)}} & \text{when } i\in J(\bar{x})\\ & \text{and } (A_id-B_id)^2+\lambda(A_id)(B_id)>0\\ \text{arbitrary} & \text{when } i\in J(\bar{x})\\ & \text{and } (A_id-B_id)^2+\lambda(A_id)(B_id)=0,\end{cases} \qquad (7)$$

$$w_i=\begin{cases}1-\frac{2(g_i(\bar{x})-f_i(\bar{x}))+\lambda f_i(\bar{x})}{2\sqrt{(g_i(\bar{x})-f_i(\bar{x}))^2+\lambda f_i(\bar{x})g_i(\bar{x})}} & \text{when } i\notin J(\bar{x})\\ 1-\frac{2(B_id-A_id)+\lambda A_id}{2\sqrt{(A_id-B_id)^2+\lambda(A_id)(B_id)}} & \text{when } i\in J(\bar{x})\\ & \text{and } (A_id-B_id)^2+\lambda(A_id)(B_id)>0\\ \text{arbitrary} & \text{when } i\in J(\bar{x})\\ & \text{and } (A_id-B_id)^2+\lambda(A_id)(B_id)=0.\end{cases}$$

Example 6. Consider the following GCP function.

$$\Phi_1(x) := \phi_p(f(x), g(x)) + \alpha f(x)_+ g(x)_+, \alpha > 0.$$

where all the operations are performed componentwise and $\phi_p(a,b) := a + b - \|(a,b)\|_p$, p is any fixed real number in the interval $(1,+\infty)$, $\|(a,b)\|_p$ denotes the p-norm of (a,b), i.e., $\|(a,b)\|_p = \sqrt[p]{|a|^p + |b|^p}$ and $a_+ = \max\{0,a\}$. This function for NCP context is studied in [2].

Now let

$$J(\bar{x}) := \{i : f_i(\bar{x}) = 0 = g_i(\bar{x})\} \quad \text{and}$$

$$K(\bar{x}) := \{i : f_i(\bar{x}) > 0, g_i(\bar{x}) > 0\}.$$

The H-differential of Φ_1 at $\bar{x}$ is given by

$$T_{\Phi_1}(\bar{x}) = \{VA + WB : (A,V,W,d) \in \Gamma\},$$

where Γ is the set of all quadruples (A,B,V,W,d) with $A \in T_f(\bar{x})$, $B \in T_g(\bar{x})$, $\|d\| = 1$, $V = \mathrm{diag}(v_i)$ and $W = \mathrm{diag}(w_i)$ are diagonal matrices with

$$v_i = \begin{cases} 1 - \dfrac{f_i(\bar{x})^{p-1}}{(f_i(\bar{x})^p + g_i(\bar{x})^p)^{\frac{p-1}{p}}} + \alpha g_i(\bar{x}) & i \in K(\bar{x}), \\ 1 - \dfrac{|A_i d|^{p-1}\mathrm{sgn}(A_i d)}{(|A_i d|^p + |B_i d|^p)^{\frac{p-1}{p}}} & i \in J(\bar{x}) \\ & \text{and } |A_i d|^p + |B_i d|^p > 0, \\ 1 - \dfrac{|f_i(\bar{x})|^{p-1}\mathrm{sgn}(f_i(\bar{x}))}{(|f_i(\bar{x})|^p + |g_i(\bar{x})|^p)^{\frac{p-1}{p}}} & i \notin J(\bar{x}) \cup K(\bar{x}), \\ \text{arbitrary} & i \in J(\bar{x}) \text{ and } |A_i d|^p + |B_i d|^p = 0, \end{cases} \tag{8}$$

$$w_i = \begin{cases} 1 - \dfrac{g_i(\bar{x})^{p-1}}{(f_i(\bar{x})^p + g_i(\bar{x})^p)^{\frac{p-1}{p}}} + \alpha f_i(\bar{x}) & i \in K(\bar{x}), \\ 1 - \dfrac{|B_i d|^{p-1}\mathrm{sgn}(B_i d)}{(|A_i d|^p + |B_i d|^p)^{\frac{p-1}{p}}} & i \in J(\bar{x}) \\ \text{and } |A_i d|^p + |B_i d|^p > 0, & \\ 1 - \dfrac{|g_i(\bar{x})|^{p-1}\mathrm{sgn}(g_i(\bar{x}))}{(|f_i(\bar{x})|^p + |g_i(\bar{x})|^p)^{\frac{p-1}{p}}} & i \notin J(\bar{x}) \cup K(\bar{x}), \\ \text{arbitrary} & i \in J(\bar{x}) \text{ and } |A_i d|^p + |B_i d|^p = 0. \end{cases} \tag{9}$$

The above calculation relies on the observation that the following is an H-differential of the one variable function $t \mapsto t_+$ at any $\bar{t}$:

$$\triangle(\bar{t}) = \begin{cases} \{1\} & \text{if } \bar{t} > 0 \\ \{0,1\} & \text{if } \bar{t} = 0 \\ \{0\} & \text{if } \bar{t} < 0. \end{cases}$$

Example 7. For the NCP function [2]

$$\phi_3(a,b) := \sqrt{[\phi_p(a,b)]^2 + \alpha(a_+ b_+)^2},$$

where $\alpha > 0$, we consider the following GCP function

$$\Phi_3(x) := \sqrt{[\phi_p(f(x), g(x))]^2 + \alpha(f(x)_+ g(x)_+)^2}, \tag{10}$$

where $\alpha > 0$, and all the operations in (10) are performed componentwise. Let

$$J(\bar{x}) := \{i : f_i(\bar{x}) = 0 = g_i(\bar{x})\} \quad \text{and}$$

$$K(\bar{x}) := \{i : f_i(\bar{x}) > 0, g_i(\bar{x}) > 0\}.$$

When $i \notin K(\bar{x})$, $(\Phi_3(\bar{x}))_i = |(\Phi_p(\bar{x}))_i| = -(\Phi_p(\bar{x}))_i$.
The H-differential of Φ_3 at $\bar{x}$ is given by

$$T_{\Phi_3}(\bar{x}) = \{VA + WB : (A,V,W,d) \in \Gamma\},$$

where Γ is the set of all quadruples (A,B,V,W,d) with $A \in T_f(\bar{x})$, $B \in T_g(\bar{x})$, $\|d\| = 1$, $V = \mathrm{diag}(v_i)$ and $W = \mathrm{diag}(w_i)$ are diagonal matrices with

$$v_i = \begin{cases} \dfrac{\phi_p(f_i(\bar{x}), g_i(\bar{x}))\left(1 - \dfrac{f_i(\bar{x})^{p-1}}{(f_i(\bar{x})^p + g_i(\bar{x})^p)^{\frac{p-1}{p}}}\right) + \alpha f_i(\bar{x}) g_i^2(\bar{x})}{\sqrt{\phi_p^2(f_i(\bar{x}), g_i(\bar{x})) + \alpha(f_i(\bar{x}), g_i(\bar{x}))^2}} & i \in K(\bar{x}), \\ \dfrac{|A_i d|^{p-1}\mathrm{sgn}(A_i d)}{(|A_i d|^p + |B_i d|^p)^{\frac{p-1}{p}}} - 1 & i \in J(\bar{x}) \\ & \text{and } |A_i d|^p + |B_i d|^p > 0, \\ \dfrac{|f_i(\bar{x})|^{p-1}\mathrm{sgn}(f_i(\bar{x}))}{(|f_i(\bar{x})|^p + |g_i(\bar{x})|^p)^{\frac{p-1}{p}}} - 1 & i \notin J(\bar{x}) \cup K(\bar{x}), \\ \text{arbitrary} & i \in J(\bar{x}) \text{ and } |A_i d|^p + |B_i d|^p = 0, \end{cases} \tag{11}$$

$$w_i = \begin{cases} \dfrac{\phi_p(f_i(\bar{x}), g_i(\bar{x}))\left(1 - \dfrac{g_i(\bar{x})^{p-1}}{(f_i(\bar{x})^p + g_i(\bar{x})^p)^{\frac{p-1}{p}}}\right) + \alpha f_i^2(\bar{x}) g_i(\bar{x})}{\sqrt{\phi_p^2(f_i(\bar{x}), g_i(\bar{x})) + \alpha(f_i(\bar{x}), g_i(\bar{x}))^2}} & i \in K(\bar{x}), \\ \dfrac{|B_i d|^{p-1}\mathrm{sgn}(B_i d)}{(|A_i d|^p + |B_i d|^p)^{\frac{p-1}{p}}} - 1 & i \in J(\bar{x}) \\ & \text{and } |A_i d|^p + |B_i d|^p > 0, \\ \dfrac{|g_i(\bar{x})|^{p-1}\mathrm{sgn}(g_i(\bar{x}))}{(|f_i(\bar{x})|^p + |g_i(\bar{x})|^p)^{\frac{p-1}{p}}} - 1 & i \notin J(\bar{x}) \cup K(\bar{x}), \\ \text{arbitrary} & i \in J(\bar{x}) \\ & \text{and } |A_i d|^p + |B_i d|^p = 0. \end{cases} \tag{12}$$

Before stating the results of this subsection, we call a vector $\bar{x}$ is said to be feasible (strictly feasible) for GCP(f,g) if $f(\bar{x}) \geq 0$ (> 0), and $g(\bar{x}) \geq 0$ (> 0). In the following theorem we will minimize the merit function under $\mathbf{E_0}(\mathbf{E})$-conditions. Since the proof of the following theorem under $\mathbf{E_0}(\mathbf{E})$-conditions will be similar to the proof of Theorem 3.5 in [24], we omit the proof.

Theorem 3. *Suppose $f, g : R^n \to R^n$ are H-differentiable at $\bar{x}$ with H-differentials, denoted by $T_f(\bar{x})$ and $T_g(\bar{x})$, respectively. Suppose Φ is a GCP function of f and g. Assume that $\Psi := \frac{1}{2}||\Phi||^2$ is H-differentiable at $\bar{x}$ with an H-differential given by*

$$\begin{aligned} T_\Psi(\bar{x}) = \{\Phi(\bar{x})^T\ [VA + WB] : A \in T_f(\bar{x}), B \in T_g(\bar{x}), \\ V = diag(v_i),\ and\ W = diag(w_i), \\ with\ v_i > 0, w_i > 0\ (\geq 0)\ whenever\ \Phi_i(\bar{x}) \neq 0\}. \end{aligned}$$

Further suppose that $\bar{x}$ is a strictly feasible point(respectively, feasible point) of GCP(f,g) and $\Phi_i(\bar{x}) > 0$, $T_g(\bar{x})$ consists of nonsingular matrices, and $S(\bar{x})$ consists of $\mathbf{E_0}(\mathbf{E})$-matrices where $S(\bar{x}) := \{AB^{-1} : A \in T_f(\bar{x}), B \in T_g(\bar{x})\}$. Then $0 \in T_\Psi(\bar{x}) \Leftrightarrow \Phi(\bar{x}) = 0$.

Concluding Remarks. This paper is considered as a generalization or an extension of [23]. In this paper, we give characterization of relatively $\mathbf{E}(\mathbf{E0})$-properties in nonsmooth functions when the underlying functions are

H-differentiable. For continuously differentiable functions, the nonsingularity of $T_g = \{\nabla g\}$ is very important from an algorithmic point of view and studying the error bounds for $\mathrm{GCP}(f,g)$

We show the usefulness of our results by giving some applications to a generalized complementarity problem corresponding to H-differentiable functions, with an associated GCP function Φ and a merit function $\Psi(x) = \frac{1}{2}||\Phi||^2$.

When the underlying functions are continuously differentiable (locally Lipschitzian, semismooth, and directionally differentiable) functions, our characterizations are valid , new characterizations for relatively $\mathbf{E}(\mathbf{E0})$-properties, and generalization to characterizations for $\mathbf{E}(\mathbf{E0})$-properties. For example, we have the following:

– When f and g are C^1 in which case $T_f(\bar{x}) = \{\nabla f(\bar{x})\}$ and $T_g(\bar{x}) = \{\nabla g(\bar{x})\}$, our results are true.
– When f is C^1 and $g(x) = x$ (in which case we can let $T_f(\bar{x}) = \{\nabla f(\bar{x})\}$), Our characterization of relatively $\mathbf{E}(\mathbf{E0})$-properties reduce to characterization of $\mathbf{E}(\mathbf{E0})$-properties in [23]. Moreover, $\mathrm{GCP}(f,g)$ reduces to nonlinear complementarity problem $\mathrm{NCP}(f)$ and the results of this paper will be valid for $\mathrm{NCP}(f)$ as applications.
– In view of Example 2, if f and g are locally Lipschitzian with $T_f(\bar{x}) = \partial f(\bar{x})$ and $T_g(\bar{x}) = \partial g(\bar{x})$, respectively, our characterizations are valid and applicable to $\mathrm{GCP}(f,g)$ when the underlying data are locally Lipschitzian.
– In view of Example 3, if f and g are semismooth (in particular, piecewise affine or piecewise smooth) with the Bouligand subdifferential $T_f(\bar{x}) = \partial_B f(\bar{x})$ and $T_g(\bar{x}) = \partial_B g(\bar{x})$, respectively, our characterizations are valid and applicable to $\mathrm{GCP}(f,g)$ when the underlying data are semismooth.

We are hoping our results will be useful to study the general variational inequalities in [14] and the iterative methods [15] for solving the general variational inequalities.

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