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Study on Existence of Periodic Solutions in a Delayed Neural Network Model

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Abstract: In this paper, we study an eight-neuron neural network model with multiple delays. By using the generalized Bendixson's criterion and second additive compound matrices, we will investigate that whether the nontrivial periodic solutions of the neural network model exist globally or not. In fact, under suitable conditions, an area will be found that contains no simple closed invariant curves including periodic orbits.

Keywords: Nontrivial periodic solutions, neural network, time delay, Bendixson's criterion

1 Introduction

The dynamical characteristics of neural networks can be applied in many sciences such as mathematics, physics and computer sciences. As time delays always occur in the signal transmission, Marcus and Westervelt proposed a neural network model with delay [1].

Dynamical behaviors such as periodic phenomenon, bifurcation and chaos have been discussed on these systems. But, since the exhaustive analysis of the dynamics of such large systems are complicated, some authors have studied the dynamical behaviors of simple systems[2,3,4].

Simplified neural networks with constant or time-varying delays have also been widely studied e.g. [5, 6,7]. Most of them focused on the local and global stability analysis. The properties of periodic solutions are also significant in many applications.

Motivated by the above, we study the global existence of periodic solutions by using the generalized Bendixson criterion. In this paper, we consider the following system:

$$\begin{cases} \dot{x_1}(t) = -\mu_1 x_1(t) + c_{11} f_{11}(y_1(t-\tau_5)) + c_{12} f_{12}(y_2(t-\tau_5)) \\ \qquad +c_{13} f_{13}(y_3(t-\tau_5)) + c_{14} f_{14}(y_4(t-\tau_5)), \\ \dot{x_2}(t) = -\mu_2 x_2(t) + c_{21} f_{21}(y_1(t-\tau_6)) + c_{22} f_{22}(y_2(t-\tau_6)) \\ \qquad +c_{23} f_{23}(y_3(t-\tau_6)) + c_{24} f_{24}(y_4(t-\tau_6)), \\ \dot{x_3}(t) = -\mu_3 x_3(t) + c_{31} f_{31}(y_1(t-\tau_7)) + c_{32} f_{32}(y_2(t-\tau_7)) \\ \qquad +c_{33} f_{33}(y_3(t-\tau_7)) + c_{34} f_{34}(y_4(t-\tau_7)), \\ \dot{x_4}(t) = -\mu_4 x_4(t) + c_{41} f_{41}(y_1(t-\tau_8)) + c_{42} f_{42}(y_2(t-\tau_8)) \\ \qquad +c_{43} f_{43}(y_3(t-\tau_8)) + c_{44} f_{44}(y_4(t-\tau_8)), \\ \dot{y_1}(t) = -\mu_5 y_1(t) + c_{51} f_{51}(x_1(t-\tau_1)) + c_{52} f_{52}(x_2(t-\tau_2)) \\ \qquad +c_{53} f_{53}(x_3(t-\tau_3)) + c_{54} f_{54}(x_4(t-\tau_4)), \\ \dot{y_2}(t) = -\mu_6 y_2(t) + c_{61} f_{61}(x_1(t-\tau_1)) + c_{62} f_{62}(x_2(t-\tau_2)) \\ \qquad +c_{63} f_{63}(x_3(t-\tau_3)) + c_{64} f_{64}(x_4(t-\tau_4)), \\ \dot{y_3}(t) = -\mu_7 y_3(t) + c_{71} f_{71}(x_1(t-\tau_1)) + c_{72} f_{72}(x_2(t-\tau_2)) \\ \qquad +c_{73} f_{73}(x_3(t-\tau_3)) + c_{74} f_{74}(x_4(t-\tau_4)), \\ \dot{y_4}(t) = -\mu_8 y_4(t) + c_{81} f_{81}(x_1(t-\tau_1)) + c_{82} f_{82}(x_2(t-\tau_2)) \\ \qquad +c_{83} f_{83}(x_3(t-\tau_3)) + c_{84} f_{84}(x_4(t-\tau_4)). \end{cases} \tag{1}$$

where c_{ij}, $(i = 1,\ldots,8;\ j = 1,\ldots,4)$ are the connection weights through the neurons in two layers: the X-layer and the Y-layer. Also, The stability of internal neuron processes on the X-layer and Y-layer have been described by $\mu_i > 0$, $(i = 1,\ldots,8)$. τ_i, $(i = 1,\ldots,8)$ correspond to the finite time delays of neural processing and delivery of signals. The activation functions have been denoted by f_{ij}, $(i = 1,\ldots,8;\ j = 1,\ldots,4)$.

We will study that whether the nontrivial periodic solutions of Eq. (1) exist globally or not by using the

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generalized Bendixson's criterion and second additive compound matrices.

This paper is organized in four sections. The next section is devoted to the definition of second additive compound matrices and generalized Bendixson's criterion. The main results on the global existence of nontrivial periodic solutions will be presented in the third section. Finally, in section 4, the conclusions will be stated.

2 preliminaries

First, we state the following proposition that will be used in our main results:

Theorem 2.1. Let $D \subset R^n$ be a simply connected region. Assume that the family of linear system

$$\dot{z}(t) = \frac{\partial f^{[2]}}{\partial x}(x(t,x_0))z(t),\ x_0 \in D \tag{2}$$

is equi-uniformly asymptotic stable. Then
(a) D contains no simple closed invariant curves including periodic orbits, homoclinic orbit, heteroclinic cycles;
(b) each semi-orbit in D converges to a single equilibrium. In particular, if D is positively invariant and contains a unique equilibrium $\bar{x}$, then $\bar{x}$ is globally asymptotically stable in D.

Proof. For the proof, see [8].

Here, $\frac{\partial f^{[2]}}{\partial x}$ is the second additive compound matrix of the Jacobian matrix $\frac{\partial f}{\partial x}$, where

$$\dot{x} = f(x),\ x \in R^n,\ f \in C^1 \tag{3}$$

is a system of ordinary differential equations.

Remark. The second additive compound matrix $A^{[2]}$ is $\binom{n}{2} \times \binom{n}{2}$, where for each $i = 1,\ldots,\binom{n}{2}$, $(i) = (i_1,i_2)$ is ith member that $1 \le i_1 < i_2 \le n$. The (i,j)-component of $A^{[2]}$ is defined as follows:
For $(i) = (j)$, the component is $a_{i_1i_1} + a_{i_2i_2}$. If just i_r does not exist in (j) and only j_s not in (i) then the component is $(-1)^{r+s}a_{i_rj_s}$, and the component is 0 if there exist no common component in (i) and (j).

For more information on the second additive compound matrix, see [9].

3 Main results

In order to establish the main results for model (1), it is necessary to make the following assumptions:
(H1) $f_{ij} \in C^k$, $f_{ij}(0) = 0\,(k = 1,2,3,\ldots,\ j = 1,\ldots,4,\ i = 1,\ldots,8)$.
(H2) $\tau_1+\tau_5 = \tau_2+\tau_6 = \tau_3+\tau_7 = \tau_4+\tau_8 = \tau$.

Let $u_1(t) = x_1(t-\tau_1)$, $u_2(t) = x_2(t-\tau_2)$, $u_3(t) = x_3(t-\tau_3)$, $u_4(t) = x_4(t-\tau_4)$, $u_5(t) = y_1(t)$, $u_6(t) = y_2(t)$, $u_7(t) = y_3(t)$ and $u_8(t) = y_4(t)$, then system (1) changes to the following equivalent form:

$$\begin{cases} \dot{u}_i(t) = -\mu_i u_i(t) + \sum_{j=1}^4 c_{ij} f_{ij}(u_{j+4}(t-\tau)),\ (i = 1,\ldots,4) \\ \dot{u}_i(t) = -\mu_i u_i(t) + \sum_{j=1}^4 c_{ij} f_{ij}(u_j(t)),\ (i = 5,\ldots,8) \end{cases} \tag{4}$$

By the hypothesis (H1), we can easily see that system (4) has a unique equilibrium $(0,0,0,0,0,0,0,0)$.

Letting $\tau = 0$ in the system (4), we have

$$\begin{cases} \dot{u}_i(t) = -\mu_i u_i(t) + \sum_{j=1}^4 c_{ij} f_{ij}(u_{j+4}(t)),\ (i = 1,\ldots,4) \\ \dot{u}_i(t) = -\mu_i u_i(t) + \sum_{j=1}^4 c_{ij} f_{ij}(u_j(t)),\ (i = 5,\ldots,8) \end{cases} \tag{5}$$

We make the following assumption on system (5):
(H3) $\exists \alpha,\beta > 0\, s.t.\ \eta(t,\alpha,\beta) < 0$
where

$$\begin{aligned} \eta(t,\alpha,\beta) = max\{&-7(\mu_1+\mu_2) + (\alpha+5)(|c_{12}||f'_{12}(u_6)| \\ &+ |c_{21}||f'_{21}(u_5)| + |c_{22}||f'_{22}(u_6)| + |c_{11}||f'_{11}(u_5)|) + \\ &(3+\alpha+\frac{1}{\beta})|c_{23}||f'_{23}(u_7)| + (4+\alpha+ \\ &\frac{1}{\beta})(|c_{24}||f'_{24}(u_8)| + |c_{13}||f'_{13}(u_7)| + |c_{14}||f'_{14}(u_8)|) \\ &+ 6(|c_{31}||f'_{31}(u_5)| + |c_{32}||f'_{32}(u_6)|) + 6(|c_{41}||f'_{41}(u_5)| \\ &+ |c_{42}||f'_{42}(u_6)|) + (5+\frac{1}{\beta})(|c_{43}||f'_{43}(u_7)| + \\ &|c_{44}||f'_{44}(u_8)|) + (5+\frac{1}{\alpha})|c_{52}||f'_{52}(u_2)| + \\ &6(|c_{53}||f'_{53}(u_3)| + |c_{54}||f'_{54}(u_4)|) + (5+\frac{1}{\alpha})|c_{62}||f'_{62}(u_2)| \\ &+ 6(|c_{63}||f'_{63}(u_3)| + |c_{64}||f'_{64}(u_4)|) + (4+\beta+ \\ &\frac{1}{\alpha})|c_{72}||f'_{72}(u_2)| + (\beta+5)(|c_{73}||f'_{73}(u_3)| + \\ &|c_{74}||f'_{74}(u_4)|) + (4+\beta+\frac{1}{\alpha})|c_{82}||f'_{82}(u_2)| + \\ &(5+\beta)(|c_{83}||f'_{83}(u_3)| + |c_{84}||f'_{84}(u_4)|) + (5+ \\ &\frac{1}{\beta})(|c_{33}||f'_{33}(u_7)| + |c_{34}||f'_{34}(u_8)|) + (5+ \\ &\frac{1}{\alpha})(|c_{51}||f'_{51}(u_1)| + |c_{61}||f'_{61}(u_1)|) + (4+\beta+ \\ &\frac{1}{\alpha})(|c_{71}||f'_{71}(u_1)| + |c_{81}||f'_{81}(u_1)|) - 7(\mu_3+\mu_4) \\ &-7(\mu_5+\mu_6) - 7(\mu_7+\mu_8)\}. \end{aligned}$$

Theorem 3.1. If the hypotheses (H1) and (H3) are satisfied, then system (5) has no nonconstant periodic solutions.

Proof. Assuming the right hand function of system (5) as $f(u(t))$, and computing the second additive compound matrix of $\frac{\partial f}{\partial u}$, we have the second compound system

$$\dot{z} = \frac{\partial f^{[2]}}{\partial u}z,\ z = (z_1,\ldots,z_{28})^T. \tag{6}$$

Let

$$W(z) = max\{\alpha|z_1|, |z_2|, |z_3|, |z_4|, |z_5|, |z_6|, |z_7|, |z_8|, |z_9|, |z_{10}|, |z_{11}|, |z_{12}|, |z_{13}|, |z_{14}|, |z_{15}|, |z_{16}|, |z_{17}|, |z_{18}|, |z_{19}|, |z_{20}|, |z_{21}|, |z_{22}|, |z_{23}|, |z_{24}|, |z_{25}|, |z_{26}|, |z_{27}|, \beta|z_{28}|\},$$

where $\alpha, \beta > 0$ are constants. Now, by using the right-hand derivative, we can prove

$$\frac{d^+}{dt} W(z(t)) \leq \eta(t, \alpha, \beta) W(z(t)) \tag{7}$$

This establishes the equi-uniform asymptotic stability of the second compound system (6). By using (H3) and Theorem 2.1 the proof is complete.

Theorem 3.2. System (4) has no nontrivial periodic solution of period τ.

Proof. It is easy to see that if $u(t)$ is a τ-periodic solution of system (4), then $u(t)$ is τ-periodic solution of the ordinary differential equation (5). So, the conclusion follows from Theorem 3.1.

4 Conclusions

In this paper, we have proved that the system (4) has no nontrivial τ-periodic solution and system (5) has no nonconstant periodic solutions. Although, our analysis was on the eight-neuron neural network model with multiple delays, the complexity found in this case might be carried over to larger scale neural networks. The proposed method in this paper, can be useful in solving problems of both theoretical and practical importance in nonlinear dynamical systems.

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