Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/100201

A Unified Version of Ran and Reuring's Theorem and Nieto and Rodríguez-López's Theorem and Low-Dimensional Generalizations

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Received: 2 Aug. 2015, Revised: 20 Oct. 2015, Accepted: 21 Oct. 2015
Published online: 1 Mar. 2016

Abstract: In this paper, we show the very close relationship between two of the pioneering theorems on fixed point theory in partially ordered metric spaces, such us Ran and Reuring's theorem and Nieto and Rodríguez-López's theorem. Although they seem to be independent, they are both two faces of an unified result. Furthermore, we extend the kind of control functions involved in the contractivity condition and we use preorders rather than partial orders, which have the main advantage of unify, in a same condition, two usual cases: the framework in which none binary relation is considered and the partially ordered case.

Keywords: Fixed point, coupled fixed point, preorder, altering distance function

1 Introduction

A *contraction* in a metric space (X,d) is a self-map $T : X \to X$ such that there exists a constant $\lambda \in [0,1)$ verifying $d(Tx,Ty) \leq \lambda\ d(x,y)$ for all $x,y \in X$. The celebrated *Banach Contractive Mapping Principle* guarantees that every contraction in a complete metric space into itself has a unique fixed point, that is, a point $x \in X$ such that $Tx = x$.

In 2004, Ran and Reuring iniciated the study of fixed point theory in metric spaces provided with a partial order.

Theorem 1(Ran and Reurings [19], Theorem 2.1). *Let $(X,\preccurlyeq)$ be an ordered set endowed with a metric d and $T : X \to X$ be a given mapping. Suppose that the following conditions hold:*

(a)(X,d) is complete.
(b)T is nondecreasing (w.r.t. $\preccurlyeq$).
(c)T is continuous.
(d)There exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$.
(e)There exists a constant $\lambda \in (0,1)$ such that $d(Tx,Ty) \leq \lambda\ d(x,y)$ for all $x,y \in X$ with $x \succcurlyeq y$.

Then T has a fixed point. Moreover, if for all $(x,y) \in X^2$ there exists $z \in X$ such that $x \preccurlyeq z$ and $y \preccurlyeq z$, we obtain uniqueness of the fixed point.

Nieto and Rodríguez-López [18] slightly modified the hypothesis of the previous result obtaining the following theorem.

Theorem 2(Nieto and Rodríguez-López [18], Theorem 2.2). *Let $(X,\preccurlyeq)$ be an ordered set endowed with a metric d and $T : X \to X$ be a given mapping. Suppose that the following conditions hold:*

(a)(X,d) is complete.
(b)T is nondecreasing (w.r.t. $\preccurlyeq$).
(c)If a nondecreasing sequence $\{x_m\}$ in X converges to a some point $x \in X$, then $x_m \preccurlyeq x$ for all m.
(d)There exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$.
(e)There exists a constant $\lambda \in (0,1)$ such that $d(Tx,Ty) \leq \lambda\ d(x,y)$ for all $x,y \in X$ with $x \succcurlyeq y$.

Then T has a fixed point. Moreover, if for all $(x,y) \in X^2$ there exists $z \in X$ such that $x \preccurlyeq z$ and $y \preccurlyeq z$, we obtain uniqueness of the fixed point.

Both theorems seem to be different because hypotheses (c) on each result are independent. If we pay attention to conditions (c) on both theorems, we may see that they are very different in the following sense: the continuity is a property on the mapping T, and the regularity is an assumption in the ordered metric space.

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Consequently, most of results proved after the appearance of Ran and Reuring's theorem and Nieto and Rodríguez-López's theorem in this field of study included two cases: either T is continuous or $(X,d,\preccurlyeq)$ is regular. For instance, in a celebrated work, Gnana Bhaskar and Lakshmikantham [2] proved the following results.

Theorem 3(Gnana Bhaskar and Lakshmikantham [2], Theorem 2.1). *Let $(X,d,\preccurlyeq)$ be an ordered space and let $F:X\times X\to X$ be a continuous mapping having the mixed $\preceq$-monotone property on X. Assume that there exists $\lambda\in[0,1)$ with*

$$d(F(x,y),F(u,v))\leq\frac{\lambda}{2}\left[\,d(x,u)+d(y,v)\,\right]$$

for all $x,y,u,v\in X$ such that $x\succcurlyeq u$ and $y\preccurlyeq v$. If there exist $x_0,y_0\in X$ such that

$$x_0\preccurlyeq F(x_0,y_0)\quad\text{and}\quad y_0\succcurlyeq F(y_0,x_0),$$

then there exist $x,y\in X$ such that

$$x=F(x,y)\quad\text{and}\quad y=F(y,x).$$

Theorem 4(Gnana Bhaskar and Lakshmikantham [2], Theorem 2.2). *Let $(X,\preccurlyeq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X,d) is a complete metric space . Suppose that X has the following property:*

(i)if a non-decreasing sequence $\{x_n\}\to x$, then $x_n\preccurlyeq x$ for all $n\in\mathbb{N}$;
(ii)if a non-increasing sequence $\{y_n\}\to y$, then $y\preccurlyeq y_n$ for all $n\in\mathbb{N}$.

Let $F:X\times X\to X$ be a mapping having the mixed $\preceq$-monotone property on X. Assume that there exists $\lambda\in[0,1)$ with

$$d(F(x,y),F(u,v))\leq\frac{\lambda}{2}\left[\,d(x,u)+d(y,v)\,\right]$$

for all $x,y,u,v\in X$ such that $x\succcurlyeq u$ and $y\preccurlyeq v$. If there exist $x_0,y_0\in X$ such that

$$x_0\preccurlyeq F(x_0,y_0)\quad\text{and}\quad y_0\succcurlyeq F(y_0,x_0),$$

then there exist $x,y\in X$ such that

$$x=F(x,y)\quad\text{and}\quad y=F(y,x).$$

Theorem 5(Gnana Bhaskar and Lakshmikantham [2], Theorem 2.4). *Adding the following condition to the hypothesis of Theorem 3:*

(C)for all $(x,y),(x^*,y^*)\in X\times X$ there exists $(z_1,z_2)\in X\times X$ that is comparable to (x,y) and (x^*,y^*);

we obtain the uniqueness of the coupled fixed point of F.

In the previous results, we can observe the necessity of distinguishing between whether T is continuous or $(X,d,\preccurlyeq)$ is regular. After the appearance of these results, the literature on coupled, tripled, quadrupled (and, even, multidimensional) fixed point theory in the setting of partially ordered metric spaces has grown exponentially. To cite some of them, we refer the reader to Gnana Bhaskar and Lakshmikantham [2], Lakshmikantham and Ćirić [3], Choudhury and Kundu [26], Berinde and Borcut [5,8], Karapınar [13], Karapınar and Luong [15], Berzig and Samet [7], Roldán *et al.* [21,22,23,24], Wang [30], Karapınar *et al.* [16], Berzig *et al.* [6], Karapınar and Agarwal [14], Roldán and Karapınar [20], Agarwal *et al.* [1] and Al-Mezel *et al.* [25], among others.

This work has three main aims. On the one hand, we present a new condition that unifies the alternative between whether T is continuous or the ordered metric space is regular. Therefore, we show that, although they are independent conditions, both assumptions are intimately related. As a consequence, from now on, researchers interested in this field of study can analyze both conditions in an unified way. On the other hand, the second objective is to relax the assumptions on the control functions involved in the contractivity condition. Thus, our results extend and unify some well known very recent results in this field. Finally, we involve preorders rather than partial orders, which have the main advantage of unifying, in a same condition, two usual cases: the framework in which none binary relation is considered and the partially ordered case.

2 Preliminaries

In the sequel, $\mathbb{N}=\{0,1,2,3,\ldots\}$ denotes the set of all nonnegative integers and $\mathbb{R}$ denotes the set of all real numbers. Henceforth, X and Y will denote nonempty sets. Elements of X are usually called *points*.

Let $T:X\to Y$ be a mapping. The *domain of* T is X and it is denoted by $\mathrm{Dom}\,T$. Its *range*, that is, the set of values of T in Y, is denoted by $T(X)$. A mapping T is completely characterized by its domain, its range, and the manner in which each *origin* $x\in\mathrm{Dom}\,T$ is applied on its *image* $T(x)\in T(X)$. For simplicity, we denote, as usual, $T(x)$ by Tx. For any set X, we denote the *identity mapping on* X by $I_X:X\to X$, which is defined by $I_Xx=x$ for all $x\in X$.

Given a self-mapping $T:X\to X$, we will say that a point $x\in X$ is a *fixed point of* T if $Tx=x$. We will denote by $\mathrm{Fix}(T)$ the set of all fixed points of T. Given a mapping $F:X\times X\to X$, a *coupled fixed point of* F is a point $(x,y)\in X$ such that $F(x,y)=x$ and $F(y,x)=y$.

Given two mappings $T:X\to Y$ and $S:Y\to Z$, the *composite of* T *and* S is the mapping $S\circ T:X\to Z$ given by

$$(S\circ T)x=STx\quad\text{for all }x\in\mathrm{Dom}\,T.$$

We say that two self-mappings $T,S:X\to X$ are *commuting* if $TSx=STx$ for all $x\in X$ (that is, $T\circ S=S\circ T$).

The *iterates* of a self-mapping $T : X \to X$ are the mappings $\{T^n : X \to X\}_{n\in\mathbb{N}}$ defined by

$$T^0 = I_X, \quad T^1 = T, \quad T^2 = T \circ T,$$
$$T^{n+1} = T \circ T^n \quad \text{for all } n \geq 2.$$

The notion of *metric space* and the concepts of *convergent sequence* and *Cauchy sequence* in a metric space can be found, for instance, in [29]. We will write $\{x_n\} \to x$ when a sequence $\{x_n\}_{n\in\mathbb{N}}$ of points of X converges to $x \in X$ in the metric space (X,d). A metric space (X,d) is *complete* if every Cauchy sequence in X converges to some point of X. The limit of a convergent sequence in a metric space is unique.

In a metric space (X,d), a mapping $T : X \to X$ is *continuous at a point* $z \in X$ if $\{Tx_n\} \to Tz$ for all sequence $\{x_n\}$ in X such that $\{x_n\} \to z$. And T is *continuous* if it is continuous at every point of X.

A *binary relation on* X is a nonempty subset $\mathscr{R}$ of $X \times X$. For simplicity, we denote $x \preceq y$ if $(x,y) \in \mathscr{R}$, and we will say that $\preceq$ is the binary relation on X. This notation let us to write $x \prec y$ when $x \preceq y$ and $x \neq y$. We write $y \succeq x$ when $x \preceq y$. A binary relation $\preceq$ on X is *reflexive* if $x \preceq x$ for all $x \in X$; it is *transitive* if $x \preceq z$ for all $x,y,z \in X$ such that $x \preceq y$ and $y \preceq z$; and it is *antisymmetric* if $x \preceq y$ and $y \preceq x$ imply $x = y$.

A reflexive and transitive relation on X is a *preorder* (or a *quasiorder*) on X. In such a case, $(X,\preceq)$ is a *preordered space*. If a preorder $\preccurlyeq$ is also antisymmetric, then $\preccurlyeq$ is called a *partial order*, and $(X,\preccurlyeq)$ is a *partially ordered space* (or a *partially ordered set*). We will use the symbol $\preceq$ for a general binary relation on X or a preorder on X, and the symbol $\preccurlyeq$ for a partial order on X.

The usual order of the set of all real numbers $\mathbb{R}$ is denoted by $\leq$. In fact, this partial order can be induced on any non-empty subset $A \subseteq \mathbb{R}$. Let $\preccurlyeq$ be the binary relation on $\mathbb{R}$ given by

$$x \preccurlyeq y \;\Leftrightarrow\; \left(x = y \quad \text{or} \quad x < y \leq 0 \right).$$

Then $\preccurlyeq$ is a partial order on $\mathbb{R}$, but it is different from $\leq$. Any equivalence relation is a preorder.

An *ordered metric space* is a triple $(X,d,\preccurlyeq)$ where (X,d) is a metric space and $\preccurlyeq$ is a partial order on X. And if $\preceq$ is a preorder on X, then $(X,d,\preceq)$ is a *preordered metric space*.

Definition 1.*Let (X,d) be a metric space, let $A \subseteq X$ be a non-empty subset and let $\preceq$ be a binary relation on X. Then $(A,d,\preceq)$ is said to be:*

– non-decreasing-regular *if for all sequence $\{x_m\} \subseteq A$ such that $\{x_m\} \to a \in A$ and $x_m \preceq x_{m+1}$ for all $m \in \mathbb{N}$, we have that $x_m \preceq a$ for all $m \in \mathbb{N}$;*
– non-increasing-regular *if for all sequence $\{x_m\} \subseteq A$ such that $\{x_m\} \to a \in A$ and $x_m \succeq x_{m+1}$ for all $m \in \mathbb{N}$, we have that $x_m \succeq a$ for all $m \in \mathbb{N}$;*
– regular *if it is both non-decreasing-regular and non-increasing-regular.*

Some authors called *ordered complete* to a regular ordered metric space (see, for instance, [10]). Furthermore, Roldán *et al.* called *sequential monotone property* to non-decreasing-regularity (see [21]).

Let $\preceq$ be a binary relation on X and let $T : X \to X$ be a mapping. We say that T is $\preceq$*-non-decreasing* if $Tx \preceq Ty$ for all $x,y \in X$ such that $x \preceq y$.

Definition 2.*Let X be a non-empty set endowed with a binary relation $\preceq$ and let $F : X^2 \to X$ be a mapping. The mapping F is said to have the* mixed $\preceq$-monotone property *if $F(x,y)$ is monotone $\preceq$-non-decreasing in x and monotone $\preceq$-non-increasing in y, that is, for all $x,y \in X$,*

$$x_1,x_2 \in X, \quad x_1 \preceq x_2 \quad \Rightarrow \quad F(x_1,y) \preceq F(x_2,y)$$

and

$$y_1,y_2 \in X, \quad y_1 \preceq y_2 \quad \Rightarrow \quad F(x,y_1) \succeq F(x,y_2).$$

3 Control functions

One of the most important ingredients in a contractivity condition is the kind of control function that are used. In this paper, we extend the control functions that were used in previous manuscripts.

Let consider the following families of control functions.

$$\begin{aligned} \mathscr{F}_{\text{alt}} = \{\, \phi : [0,\infty) \to [0,\infty) :\ & \phi \text{ continuous}, \\ & \text{non-decreasing}, \ \phi(t) = 0 \Leftrightarrow t = 0 \,\}, \\ \mathscr{F}'_{\text{alt}} = \{\, \phi : [0,\infty) \to [0,\infty) :\ & \phi \text{ lower semi-continuous}, \\ & \phi(t) = 0 \Leftrightarrow t = 0 \,\}. \end{aligned}$$

Functions in $\mathscr{F}_{\text{alt}}$ are called *altering distance functions* (see [17,6,20,22]). To extend the previous kind of functions, we will also consider:

$$\begin{aligned} \mathscr{F} = \{\, \phi : [0,\infty) \to [0,\infty) :\ & \phi \text{ continuous from the right}, \\ & \text{non-decreasing}, \ \phi(t) = 0 \Leftrightarrow t = 0 \,\}. \end{aligned}$$

Clearly, $\mathscr{F}_{\text{alt}} \subset \mathscr{F}$.

*Example 1.*The function $\psi : [0,\infty) \to [0,\infty)$, defined by

$$\psi(t) = \begin{cases} t, & \text{if } 0 \leq t < 1, \\ 2t, & \text{if } t \geq 1, \end{cases}$$

belongs to $\mathscr{F}$ but it is not an altering distance function.

Under the non-decreasingness assumption, upper semi-continuity implies right-continuity.

Lemma 1.*If $\psi : [0,\infty) \to [0,\infty)$ is non-decreasing and upper semi-continuous, then ψ is continuous from the right.*

Proof. To prove it, let $s_0 \geq 0$ be arbitrary. The upper semi-continuity of ψ in s_0 means that

$$\psi(s_0) \geq \limsup_{t \to s_0} \psi(t).$$

Let $\{t_n\}$ be an arbitrary strictly decreasing sequence converging to s_0. Therefore, $s_0 < t_{n+1} < t_n$ for all $n \in \mathbb{N}$. As ψ is non-decreasing, $\psi(s_0) \leq \psi(t_{n+1}) \leq \psi(t_n)$ for all $n \in \mathbb{N}$, which means that $\{\psi(t_n)\}$ is a bounded below, non-increasing sequence. As a consequence, it is convergent and

$$\psi(s_0) \leq \lim_{n \to \infty} \psi(t_n).$$

Therefore,

$$\psi(s_0) \leq \lim_{n \to \infty} \psi(t_n) \leq \limsup_{t \to s_0} \psi(t) \leq \psi(s_0),$$

which means that

$$\lim_{n \to \infty} \psi(t_n) = \psi(s_0).$$

As a consequence, ψ is continuous at s_0 from the right.

The following properties are well known using functions on $\mathscr{F}_{\text{alt}}$, but they are also valid in $\mathscr{F}$.

Lemma 2. *Let $\psi, \phi : [0,\infty) \to [0,\infty)$ be two functions such that ψ is non-decreasing and $\phi^{-1}(\{0\}) = \{0\}$, and let $t,s,r \in [0,\infty)$.*

1. *If $\psi(t) \leq \psi(s) - \phi(r)$, then $t < s$ or $r = 0$.*
2. *If ψ also verifies $\psi^{-1}(\{0\}) = \{0\}$ and $\psi(t) \leq (\psi - \phi)(s)$, then $t < s$ or $t = s = 0$.*

Proof. **(1)** Assume that $t \geq s$ and we have to prove that $r = 0$. Indeed, as ψ is non-decreasing, $\psi(s) \leq \psi(t)$. Therefore, $\psi(t) \leq \psi(s) - \phi(r) \leq \psi(s) \leq \psi(t)$. As a consequence, $\psi(t) = \psi(s)$ and $\phi(r) = 0$. Therefore $r = 0$.

(2) Next, assume that $\psi(t) \leq (\psi - \phi)(s)$ and $t \geq s$. By item (1), $s = 0$. Therefore, $0 \leq \psi(t) \leq \psi(0) - \phi(0) = 0$, so $\psi(t) = 0$ and $t = 0$.

Lemma 3. *Let $\psi \in \mathscr{F}$, $\phi \in \mathscr{F}'_{\text{alt}}$ and let $\{t_n\} \subset [0,\infty)$ be a sequence such that*

$$\psi(t_{n+1}) \leq \psi(t_n) - \phi(t_n) \quad \text{for all } n \in \mathbb{N}. \tag{1}$$

Then $\{t_n\} \to 0$.

Proof. We distinguish two cases.

Case 1. There exists $n_0 \in \mathbb{N}$ such that $t_{n_0} \leq t_{n_0+1}$. In this case, as ψ is non-decreasing, we have that $\psi(t_{n_0}) \leq \psi(t_{n_0+1}) \leq \psi(t_{n_0}) - \phi(t_{n_0}) \leq \psi(t_{n_0})$. Therefore, $\psi(t_{n_0}) = \psi(t_{n_0+1})$ and $\phi(t_{n_0}) = 0$. As $\phi \in \mathscr{F}'_{\text{alt}}$, then $t_{n_0} = 0$. Moreover, $0 \leq \psi(t_{n_0+1}) \leq \psi(t_{n_0}) - \phi(t_{n_0}) = \psi(0) - \phi(0) = 0$, so $\psi(t_{n_0+1}) = 0$ and also $t_{n_0+1} = 0$. By induction, we can show that $t_n = 0$ for all $n \geq n_0$. In particular, $\{t_n\} \to 0$.

Case 2. $t_{n+1} < t_n$ for all $n \in \mathbb{N}$. In this case, $\{t_n\}$ is a bounded below, strictly decreasing sequence. Then, it is convergent. Let $L \in [0,\infty)$ be its limit. Then $L < t_{n+1} < t_n$ for all $n \in \mathbb{N}$. As ψ is continuous from the right,

$$\psi(L) = \lim_{n \to \infty} \psi(t_n).$$

By (1),

$$0 \leq \phi(t_n) \leq \psi(t_n) - \psi(t_{n+1}) \quad \text{for all } n \in \mathbb{N}.$$

Letting $n \to \infty$, we deduce that $\{\phi(t_n)\} \to 0$. But, as ϕ is lower semi-continuous,

$$0 \leq \phi(L) \leq \liminf_{t \to L} \phi(t) \leq \lim_{n \to \infty} \phi(t_n) = 0.$$

Hence, $\phi(L) = 0$ and $L = 0$.

Lemma 4. *Let $\{t_n\}, \{s_n\} \subset [0,\infty)$ be two sequences that converge to the same limit $L \in [0,\infty)$. Assume that $L < t_n$ for all $n \in \mathbb{N}$ and there exist two functions $\psi \in \mathscr{F}$ and $\phi \in \mathscr{F}'_{\text{alt}}$ such that*

$$\psi(t_n) \leq \psi(s_n) - \phi(s_n) \quad \text{for all } n \in \mathbb{N}. \tag{2}$$

Then $L = 0$.

Proof. By item 2 of Lemma 2, for all $n \in \mathbb{N}$,

$$t_n < s_n \quad \text{or} \quad t_n = s_n = 0.$$

The second case is impossible because $0 \leq L < t_n$ for all $n \in \mathbb{N}$. As a consequence,

$$L < t_n < s_n \quad \text{for all } n \in \mathbb{N}.$$

As ψ is right-continuous at L, we deduce that

$$\psi(L) = \lim_{n \to \infty} \psi(t_n) = \lim_{n \to \infty} \psi(s_n).$$

By (2),

$$0 \leq \phi(s_n) \leq \psi(s_n) - \psi(t_{n+1}) \quad \text{for all } n \in \mathbb{N}.$$

Letting $n \to \infty$, we deduce that $\{\phi(s_n)\} \to 0$. But, as ϕ is lower semi-continuous,

$$0 \leq \phi(L) \leq \liminf_{t \to L} \phi(t) \leq \lim_{n \to \infty} \phi(s_n) = 0.$$

Hence, $\phi(L) = 0$ and $L = 0$.

4 Main results

We introduce the following notion, which will be the common condition of Theorems 1 and 2.

Definition 3. *Given a metric space (X,d) endowed with a binary relation $\preceq$, a mapping $T : X \to X$ is* $(d,\preceq)$-nondecreasing-continuous at $z_0 \in X$ *if we have that $\{Tx_m\}$ converges to Tz_0 for all $\preceq$-nondecreasing sequence $\{x_m\}$ convergent to z_0. And T is* $(d,\preccurlyeq)$-nondecreasing-continuous *if it is $(d,\preccurlyeq)$-nondecreasing-continuous at every point of X.*

It is obvious that every continuous mapping is also non-decreasing-continuous, but the converse is false.

Example 2. If $\mathbb{R}$ is endowed with the Euclidean metric ($d_e(x,y) = |x-y|$ for all $x,y \in \mathbb{R}$) and its usual partial order $\leq$, then the mapping

$$Tx = \begin{cases} 0, & \text{if } x \leq 0, \\ 1, & \text{if } x > 0, \end{cases}$$

is $(d_e, \leq)$-non-decreasing-continuous on $\mathbb{R}$, but it is not continuous at $x = 0$.

The main result in this paper is the following one.

Theorem 6. *Let $(X,d,\preceq)$ be a preordered metric space and let $T : X \to X$ be a given mapping. Suppose that the following conditions hold:*

(a) (X,d) is complete.
(b) T is nondecreasing (w.r.t. $\preceq$).
(c) T is $(d,\preceq)$-nondecreasing-continuous.
(d) There exists $x_0 \in X$ such that $x_0 \preceq Tx_0$.
(e) There exist functions $\psi \in \mathscr{F}$ and $\phi \in \mathscr{F}'_{\text{alt}}$ such that

$$\psi(d(Tx,Ty)) \leq \psi(d(x,y)) - \phi(d(x,y)) \quad (3)$$

for all $x,y \in X$ with $x \succeq y$.

Then T has a fixed point.
Furthermore, if the following assumption is satisfied:

(U) for all $x,y \in \mathrm{Fix}(T)$ there exists $z \in X$ such that z is $\preceq$-comparable, at the same time, to x and to y;

then we obtain uniqueness of the fixed point.

Proof. Let $x_0 \in X$ be any point such that $x_0 \preceq Tx_0$ and let $\{x_m\}$ the Picard sequence of T, that is,

$$x_{m+1} = Tx_m, \quad \text{for all } m \geq 0.$$

Taking into account that T is a $\preceq$-non-decreasing mapping, we observe that

$$x_0 \preceq Tx_0 = x_1 \quad \text{implies} \quad x_1 = Tx_0 \preceq Tx_1 = x_2.$$

Inductively, we obtain

$$x_0 \preceq x_1 \preceq x_2 \preceq \ldots \preceq x_{m-1} \preceq x_m \preceq x_{m+1} \preceq \ldots \quad (4)$$

If there exists m_0 such that $x_{m_0} = x_{m_0+1}$, then $x_{m_0} = x_{m_0+1} = Tx_{m_0}$, that is, T has a fixed point, which completes the existence part of the proof. On the contrary case, assume that $x_m \neq x_{m+1}$ for all $m \in \mathbb{N}$, that is, $d(x_m, x_{m+1}) > 0$ for all $m \geq 0$. Regarding (4), we set $x = x_m$ and $y = x_{m+1}$ in (3). Then we get, for all $m \in \mathbb{N}$,

$$\begin{aligned}\psi(d(x_{m+1},x_{m+2})) &= \psi(d(Tx_m,Tx_{m+1})) \\ &\leq \psi(d(x_m,x_{m+1})) - \phi(d(x_m,x_{m+1})).\end{aligned}$$

By Lemma 3, we deduce that

$$\lim_{m\to\infty} d(x_m,x_{m+1}) = 0.$$

Next, we will prove that $\{x_m\}$ is a Cauchy sequence in (X,d) reasoning by contradiction. Suppose that $\{x_m\}$ is not Cauchy. Then, following a classical argument (see, for instance, [22]), there exists a positive real number $\varepsilon_0 > 0$ and two subsequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of $\{gx_m\}$ such that, for all $k \in \mathbb{N}$,

$$\begin{aligned}&k \leq n(k) < m(k) < n(k+1), \\ &d(x_{n(k)},x_{m(k)-1}) \leq \varepsilon_0 < d(x_{n(k)},x_{m(k)})\end{aligned}$$

and also

$$\lim_{k\to\infty} d(x_{n(k)},x_{m(k)}) = \lim_{k\to\infty} d(x_{n(k)-1},x_{m(k)-1}) = \varepsilon_0. \quad (5)$$

Notice that as $\preceq$ is transitive, then $x_{n(k)-1} \preceq x_{m(k)-1}$ for all $k \in \mathbb{N}$. Using the contractivity condition (3), for all $k \in \mathbb{N}$,

$$\begin{aligned}\psi(d(x_{n(k)},x_{m(k)})) &= \psi(d(Tx_{n(k)-1},Tx_{m(k)-1})) \\ &\leq \psi(d(x_{n(k)-1},x_{m(k)-1})) - \phi(d(x_{n(k)-1},x_{m(k)-1})).\end{aligned}$$

By (5), $\{t_k = d(x_{n(k)},x_{m(k)})\}_{k\in\mathbb{N}}$ and $\{s_k = d(x_{n(k)-1},x_{m(k)-1})\}_{k\in\mathbb{N}}$ are two sequences in $[0,\infty)$ converging to the same limit $L = \varepsilon_0$ and verifying

$$\psi(t_k) \leq \psi(s_k) - \phi(s_k) \quad \text{for all } k \in \mathbb{N}.$$

Then, it follows from Lemma 4 that $\varepsilon_0 = 0$, which is a contradiction. As a consequence, we must admit that $\{x_m\}$ is a Cauchy sequence in (X,d).

Taking into account that (X,d) is complete, there exists $z \in X$ such that $\{x_m\} \to z$. In addition to this, as $\{x_m\}$ is a convergent non-decreasing sequence, and T is $(d,\preceq)$-nondecreasing-continuous, it follows that

$$\{Tx_m\} \to Tz.$$

But as $\{Tx_m = x_{m+1}\} \to z$, the uniqueness of the limit of a convergent sequence in a metric space guarantees that $Tz = z$, that is, z is a fixed point of T.

Next, to prove the uniqueness of the fixed point, let x and y be two arbitrary fixed points of T. By hypothesis (U), there exists $z \in X$ such that z is $\preceq$-comparable, at the same time, to x and to y. Let $\{z_m\}$ be the Picard sequence of T based on z, that is, $z_0 = z$ and $z_{m+1} = Tz_m$ for all $m \in \mathbb{N}$. We are going to show that $\{z_m\} \to x$ and $\{z_m\} \to y$ and, by the uniqueness of the limit, we will conclude that $x = y$. Indeed, as z is $\preceq$-comparable to x, then $z \preceq x$ or $z \succeq x$. Assume that $z \preceq x$ (the other case is similar). As T is $\preceq$-non-decreasing,

$$z_0 = z \preceq x \quad \text{implies} \quad z_1 = Tz_0 \preceq Tx = x.$$

By induction, it is possible to deduce that $z_m \preceq x$ for all $m \in \mathbb{N}$. Then, applying the contractivity condition (3), we deduce that, for all $m \in \mathbb{N}$,

$$\begin{aligned}\psi(d(x,z_{m+1})) &= \psi(d(Tx,Tz_m)) \\ &\leq \psi(d(x,z_m)) - \phi(d(x,z_m)).\end{aligned}$$

Again by Lemma 3, we deduce that

$$\lim_{m\to\infty} d(x, z_{m+1}) = 0,$$

so $\{z_m\} \to x$. If we have supposed that $x \preceq z$, the same argument would have shown that $\{z_m\} \to x$. Similarly, it can be proved that, in any case, $\{z_m\} \to y$. As a consequence, $x = y$ and T has a unique fixed point.

Remark. Following Boyd and Wong [9], the functions ψ and ϕ verifying (3) have only to be defined on $\overline{P}$, where $P = \{d(x,y) : x, y \in X\} \subseteq [0,\infty)$ is the range of d. Thus, if the metric space (X,d) is bounded, that is, if there exists $M > 0$ such that $d(x,y) \leq M$ for all $x, y \in X$, then it is not necessary to consider ψ and ϕ defined on the whole interval $[0,\infty)$. In this case, the function $\psi : [0, M+1) \to [0,\infty)$ defined by $\psi(t) = t/(M+1-t)$ for all $t \in [0, M+1)$, which cannot be continuously extended to $t = M+1$, can be useful to apply Theorem 6.

As $\mathscr{F}_{\text{alt}} \subset \mathscr{F}$, the following result is immediate.

Corollary 1. *Let $(X,d,\preceq)$ be a preordered metric space and let $T : X \to X$ be a given mapping. Suppose that the following conditions hold:*

(a) (X,d) is complete.
(b) T is nondecreasing (w.r.t. $\preceq$).
(c) T is $(d,\preceq)$-nondecreasing-continuous.
(d) There exists $x_0 \in X$ such that $x_0 \preceq Tx_0$.
(e) There exist functions $\psi \in \mathscr{F}$ and $\phi \in \mathscr{F}'_{\text{alt}}$ such that, for all $x, y \in X$ with $x \succeq y$,

$$\psi(d(Tx,Ty)) \leq \psi(d(x,y)) - \phi(d(x,y))$$

Then T has a fixed point. Furthermore, if the following assumption is satisfied:

(U) for all $x, y \in \mathrm{Fix}(T)$ there exists $z \in X$ such that z is $\preceq$-comparable, at the same time, to x and to y;

then we obtain uniqueness of the fixed point.

The following result follows from Lemma 1.

Corollary 2. *Let $(X,d,\preceq)$ be a preordered metric space and let $T : X \to X$ be a given mapping. Suppose that the following conditions hold:*

(a) (X,d) is complete.
(b) T is nondecreasing (w.r.t. $\preceq$).
(c) T is $(d,\preceq)$-nondecreasing-continuous.
(d) There exists $x_0 \in X$ such that $x_0 \preceq Tx_0$.
(e) There exists a non-decreasing, upper-semicontinuous function $\psi : [0,\infty) \to [0,\infty)$ such that $\psi^{-1}(\{0\}) = \{0\}$ and a function $\phi \in \mathscr{F}'_{\text{alt}}$ verifying, for all $x, y \in X$ with $x \succeq y$,

$$\psi(d(Tx,Ty)) \leq \psi(d(x,y)) - \phi(d(x,y))$$

Then T has a fixed point. Furthermore, if the following assumption is satisfied:

(U) for all $x, y \in \mathrm{Fix}(T)$ there exists $z \in X$ such that z is $\preceq$-comparable, at the same time, to x and to y;

then we obtain uniqueness of the fixed point.

If we take $\psi(t) = t$ for all $t \in [0,\infty)$, it follows that $\psi \in \mathscr{F}_{\text{alt}}$ and we have the following particular case.

Corollary 3. *Let $(X,d,\preceq)$ be a preordered metric space and let $T : X \to X$ be a given mapping. Suppose that the following conditions hold:*

(a) (X,d) is complete.
(b) T is nondecreasing (w.r.t. $\preceq$).
(c) T is $(d,\preceq)$-nondecreasing-continuous.
(d) There exists $x_0 \in X$ such that $x_0 \preceq Tx_0$.
(e) There exists a function $\phi \in \mathscr{F}'_{\text{alt}}$ such that, for all $x, y \in X$ with $x \succeq y$,

$$d(Tx,Ty) \leq d(x,y) - \phi(d(x,y))$$

Then T has a fixed point. Furthermore, if the following assumption is satisfied:

(U) for all $x, y \in \mathrm{Fix}(T)$ there exists $z \in X$ such that z is $\preceq$-comparable, at the same time, to x and to y;

then we obtain uniqueness of the fixed point.

The previous corollary is also valid if we assume that ϕ is a continuous function. In addition to this, if $\lambda \in [0,1)$ and we consider $\phi(t) = \lambda t$ for all $t \in [0,\infty)$, it follows that $\phi \in \mathscr{F}'_{\text{alt}}$ and we have the following particular case.

Corollary 4. *Let $(X,d,\preceq)$ be a preordered metric space and let $T : X \to X$ be a given mapping. Suppose that the following conditions hold:*

(a) (X,d) is complete.
(b) T is nondecreasing (w.r.t. $\preceq$).
(c) T is $(d,\preceq)$-nondecreasing-continuous.
(d) There exists $x_0 \in X$ such that $x_0 \preceq Tx_0$.
(e) There exists a constant $\lambda \in [0,1)$ such that, for all $x, y \in X$ with $x \succeq y$,

$$d(Tx,Ty) \leq \lambda\, d(x,y)$$

Then T has a fixed point. Furthermore, if the following assumption is satisfied:

(U) for all $x, y \in \mathrm{Fix}(T)$ there exists $z \in X$ such that z is $\preceq$-comparable, at the same time, to x and to y;

then we obtain uniqueness of the fixed point.

Next we deduce that Theorems 1 and 2 are simple consequences of our main result.

Theorem 7. *Theorem 1 follows from Theorem 6.*

Proof. It follows from the fact that if T is continuous, then it is also $(d,\preceq)$-nondecreasing-continuous, and Corollary 4 is applicable.

Theorem 8.*Theorem 2 follows from Theorem 6.*

*Proof.*We have to prove that, under all conditions of Theorem 2, T is $\preceq$-nondecreasing-continuous. Indeed, let $\{x_m\} \subseteq X$ be a $\preceq$-nondecreasing convergent sequence in X, and let $u \in X$ be its limit. By hypothesis (c) in Theorem 2, we have that $x_m \preceq u$ for all m. Applying the contractivity condition, we have that

$$d(Tx_m, Tu) \leq \lambda\, d(x_n, u) \quad \text{for all } n \in \mathbb{N}.$$

Then $\{Tx_m\} \to Tu$, and this proves that T is $(d, \preceq)$-nondecreasing-continuous. Therefore, Corollary 4 is applicable.

The reader may notice that the continuity of T guarantees the $\preceq$-non-decreasing continuity of T. However, although T is $\preceq$-non-decreasing, the regularity of $(X,d,\preceq)$ does not imply that T is $\preceq$-non-decreasing continuous. In fact, the contractivity condition plays a key role in this proof.

One of the main advantages of preorders versus partial orders is that the following two corollaries, with their respective consequences (as before) follow immediately from Theorem 6. In the following result, we involve a partial order $\preccurlyeq$.

Corollary 5(Harjani and Sadarangani [12]). *Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X,d) is a complete metric space. Let $T : X \to X$ be a $\preccurlyeq$-non-decreasing mapping such that*

$$\psi\left(d\left(Tx, Ty\right)\right) \leq \psi\left(d\left(x,y\right)\right) - \phi\left(d\left(x,y\right)\right) \quad \textit{for all } x \succcurlyeq y,$$

where ψ and ϕ are altering distance functions. Also assume that, at least, one of the following conditions holds.

(i)T is continuous, or
(ii)if a $\preccurlyeq$-non-decreasing sequence $\{x_n\}$ in X converges to a some point $x \in X$, then $x_n \preccurlyeq x$ for all n.

If there exists $x_0 \in X$ with $x_0 \preccurlyeq Tx_0$, then T has a fixed point.

Furthermore, if for all $x,y \in X$ there exists $z \in X$ such that z is $\preccurlyeq$-comparable, at the same time, to x and to y, then we obtain uniqueness of the fixed point.

In the following statement we use the fact that the binary relation $\preceq_0$ on X, defined by

$$x \preceq_0 y \quad \text{for all } x,y \in X \tag{6}$$

is a preorder on X.

Corollary 6.*Let (X,d) be a complete metric space and let $T : X \to X$ be a given mapping. Suppose that there exist functions $\psi \in \mathscr{F}$ and $\phi \in \mathscr{F}'_{\text{alt}}$ such that*

$$\psi\left(d\left(Tx, Ty\right)\right) \leq \psi\left(d\left(x,y\right)\right) - \phi\left(d\left(x,y\right)\right) \quad \textit{for all } x,y \in X.$$

Then T has a unique fixed point.

Theorem 9(Dutta and Choudhury [11], Theorem 2.1). *Let (X,d) be a complete metric space and let $T : X \to X$ be a self-mapping satisfying the inequality*

$$\psi(d(Tx,Ty)) \leq \psi(d(x,y)) - \varphi(d(x,y))$$

for all $x,y \in X$, where $\psi, \varphi : [0,\infty[\to \infty$ are both continuous and monotone nondecreasing functions with $\psi(t) = \varphi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.

5 Coupled fixed point theorems

In this section, we extend Theorem 6 to the coupled case. The following result will be useful in the proof of the main result of this section.

Lemma 5.*Let $\{x_n\},\{y_n\} \subseteq X$ be two sequences on a metric space (X,d) such that*

$$\lim_{n\to\infty} d\left(x_n, x_{n+1}\right) = \lim_{n\to\infty} d\left(y_n, y_{n+1}\right) = 0.$$

Suppose that, at least, one of them is not Cauchy in (X,d). Then there exist $\varepsilon_0 > 0$ and two sequences of natural numbers $\{n(k)\}_{k\in\mathbb{N}}$ and $\{m(k)\}_{k\in\mathbb{N}}$ such that, for all $k \in \mathbb{N}$,

$$\begin{aligned} &k \leq n(k) < m(k) < n(k+1), \\ &\max\left\{d(x_{n(k)}, x_{m(k)-1}), d(y_{n(k)}, y_{m(k)-1})\right\} \leq \varepsilon_0 \\ &\qquad < \max\left\{d(x_{n(k)}, x_{m(k)}), d(y_{n(k)}, y_{m(k)})\right\} \end{aligned}$$

and also

$$\begin{aligned} &\lim_{k\to\infty} \left[\max\left\{d(x_{n(k)}, x_{m(k)}), d(y_{n(k)}, y_{m(k)})\right\}\right] \\ &= \lim_{k\to\infty} \left[\max\left\{d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})\right\}\right] \\ &= \varepsilon_0. \end{aligned}$$

Theorem 10.*Let $(X,d,\preceq)$ be a preordered metric space and let $F : X \times X \to X$ be a given mapping. Suppose that the following conditions hold:*

(a)(X,d) is complete.
(b)F has the mixed $\preceq$-monotone property.
(c)If $z, \omega \in X$ are two points and $\{x_n\},\{y_n\} \subseteq X$ are two sequences such that $\{x_n\} \to z$, $\{y_n\} \to \omega$, and $x_n \preceq x_{n+1}$ and $y_n \succeq y_{n+1}$ for all $n \in \mathbb{N}$, then $\{F(x_n,y_n)\} \to F(z,\omega)$ and $\{F(y_n,x_n)\} \to F(\omega,z)$.
(d)There exist $x_0,y_0 \in X$ such that $x_0 \preceq F(x_0,y_0)$ and $y_0 \succeq F(y_0,x_0)$.
(e)There exist functions $\psi \in \mathscr{F}$ and $\phi \in \mathscr{F}'_{\text{alt}}$ such that

$$\begin{aligned} \psi\left(d\left(F\left(x,y\right), F\left(u,v\right)\right)\right) \leq \psi\left(\max\left\{d\left(x,u\right), d\left(y,v\right)\right\}\right) \\ - \phi\left(\max\left\{d\left(x,u\right), d\left(y,v\right)\right\}\right) \end{aligned} \tag{7}$$

for all $x,y,u,v \in X$ with $x \preceq u$ and $y \succeq v$.

Then F has, at least, a coupled fixed point.

*Proof.*Starting from the points $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$, we consider the sequences $\{x_n\}$ and $\{y_n\}$ defined by

$$x_{n+1} = F(x_n, y_n) \quad \text{and} \quad y_{n+1} = F(y_n, x_n) \quad \text{for all } n \in \mathbb{N}. \tag{8}$$

Using that F has the mixed $\preceq$-monotone property, it is possible to prove that

$$x_n \preceq x_{n+1} \quad \text{and} \quad y_n \succeq y_{n+1} \quad \text{for all } n \in \mathbb{N}.$$

Then, applying the contractivity condition (7), it follows that

$$\begin{aligned}\psi(d(x_{n+1}, x_{n+2})) &= \psi(d(F(x_n, y_n), F(x_{n+1}, y_{n+1}))) \\ &\leq \psi(\max\{d(x_n, x_{n+1}), d(y_n, y_{n+1})\}) \\ &\quad - \phi(\max\{d(x_n, x_{n+1}), d(y_n, y_{n+1})\})\end{aligned}$$

and, similarly, using that $y_{n+1} \preceq y_n$ and $x_{n+1} \succeq x_n$,

$$\begin{aligned}\psi(d(y_{n+1}, y_{n+2})) &= \psi(d(F(y_n, x_n), F(y_{n+1}, x_{n+1}))) \\ &= \psi(d(F(y_{n+1}, x_{n+1}), F(y_n, x_n))) \\ &\leq \psi(\max\{d(y_{n+1}, y_n), d(x_{n+1}, x_n)\}) \\ &\quad - \phi(\max\{d(y_{n+1}, y_n), d(x_{n+1}, x_n)\}).\end{aligned}$$

Therefore, as ψ is non-decreasing,

$$\begin{aligned}&\psi(\max\{d(x_{n+1}, x_{n+2}), d(y_{n+1}, y_{n+2})\}) \\ &\quad = \max\{\psi(d(x_{n+1}, x_{n+2})), \psi(d(y_{n+1}, y_{n+2}))\} \\ &\quad \leq \psi(\max\{d(x_n, x_{n+1}), d(y_n, y_{n+1})\}) \\ &\qquad - \phi(\max\{d(x_n, x_{n+1}), d(y_n, y_{n+1})\}).\end{aligned}$$

By Lemma 3, we deduce that

$$\lim_{n\to\infty} \max\{d(x_n, x_{n+1}), d(y_n, y_{n+1})\} = 0,$$

that is,

$$\{d(x_n, x_{n+1})\} \to 0 \quad \text{and} \quad \{d(y_n, y_{n+1})\} \to 0.$$

Next, we show that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences on (X, d) reasoning by contradiction. If, at least, one of them is not a Cauchy sequence, Lemma 5 guarantees that there exist $\varepsilon_0 > 0$ and two sequences of natural numbers $\{n(k)\}_{k\in\mathbb{N}}$ and $\{m(k)\}_{k\in\mathbb{N}}$ such that, for all $k \in \mathbb{N}$,

$$\begin{aligned}&k \leq n(k) < m(k) < n(k+1), \\ &\max\left\{d(x_{n(k)}, x_{m(k)-1}), d(y_{n(k)}, y_{m(k)-1})\right\} \leq \varepsilon_0 \\ &\quad < \max\left\{d(x_{n(k)}, x_{m(k)}), d(y_{n(k)}, y_{m(k)})\right\},\end{aligned} \tag{9}$$

and also

$$\begin{aligned}&\lim_{k\to\infty} \left[\max\left\{d(x_{n(k)}, x_{m(k)}), d(y_{n(k)}, y_{m(k)})\right\}\right] \\ &\quad = \lim_{k\to\infty} \left[\max\left\{d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})\right\}\right] \\ &\quad = \varepsilon_0.\end{aligned} \tag{10}$$

As $\preceq$ is transitive, we have that $x_{n(k)-1} \preceq x_{m(k)-1}$ and $y_{n(k)-1} \succeq y_{m(k)-1}$ for all $k \in \mathbb{N}$. Then, applying the contractivity condition (7), it follows that, for all $k \in \mathbb{N}$,

$$\begin{aligned}&\psi(d(x_{n(k)}, x_{m(k)})) \\ &\quad = \psi(d(F(x_{n(k)-1}, y_{n(k)-1}), F(x_{m(k)-1}, y_{m(k)-1}))) \\ &\quad \leq \psi(\max\{d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})\}) \\ &\qquad - \phi(\max\{d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})\}).\end{aligned}$$

Similarly, as $y_{m(k)-1} \preceq y_{n(k)-1}$ and $x_{m(k)-1} \succeq x_{n(k)-1}$ for all $k \in \mathbb{N}$, we have that

$$\begin{aligned}&\psi(d(y_{n(k)}, y_{m(k)})) = \psi(d(y_{m(k)}, y_{n(k)})) \\ &\quad = \psi(d(F(y_{m(k)-1}, x_{m(k)-1}), F(y_{n(k)-1}, x_{n(k)-1}))) \\ &\quad \leq \psi(\max\{d(y_{m(k)-1}, y_{n(k)-1}), d(x_{m(k)-1}, x_{n(k)-1})\}) \\ &\qquad - \phi(\max\{d(y_{m(k)-1}, y_{n(k)-1}), d(x_{m(k)-1}, x_{n(k)-1})\}) \\ &\quad = \psi(\max\{d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})\}) \\ &\qquad - \phi(\max\{d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})\}).\end{aligned}$$

As ψ is non-decreasing, it yields

$$\begin{aligned}&\psi(\max\{d(x_{n(k)}, x_{m(k)}), d(y_{n(k)}, y_{m(k)})\}) \\ &\quad = \max\{\psi(d(x_{n(k)}, x_{m(k)})), \psi(d(y_{n(k)}, y_{m(k)}))\} \\ &\quad \leq \psi(\max\{d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})\}) \\ &\qquad - \phi(\max\{d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})\}).\end{aligned}$$

If we consider the sequences

$$\begin{aligned}&\{t_k = \max\{d(x_{n(k)}, x_{m(k)}), d(y_{n(k)}, y_{m(k)})\}\}_{k\in\mathbb{N}} \quad \text{and} \\ &\{s_k = \max\{d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})\}\}_{k\in\mathbb{N}},\end{aligned}$$

we have proved that $\psi(t_k) \leq \psi(s_k) - \phi(s_k)$ for all $k \in \mathbb{N}$. Moreover, by (10), $\{t_k\}$ and $\{s_k\}$ converge to the same limit, which is $L = \varepsilon_0$, and by (9), $L = \varepsilon_0 < t_k$ for all $k \in \mathbb{N}$. Using Lemma 4, we conclude that $\varepsilon_0 = L = 0$, which is a contradiction. Then, we must accept that both sequences, $\{x_n\}$ and $\{y_n\}$, are Cauchy sequences in (X, d). As (X, d) is complete, there exists $z, \omega \in X$ such that $\{x_n\} \to z$ and $\{y_n\} \to \omega$. Let show that (z, ω) is a coupled fixed point of F.

As $\{x_n\}$ is $\preceq$-non-decreasing and converges to z, and $\{y_n\}$ is $\preceq$-non-increasing and converges to ω, assumption (c) guarantees that $\{F(x_n, y_n)\} \to F(z, \omega)$ and $\{F(y_n, x_n)\} \to F(\omega, z)$. Then $\{x_{n+1}\} \to F(z, \omega)$ and $\{y_{n+1}\} \to F(\omega, z)$ by (8). As the limit of a convergent sequence in a metric space is unique, then $F(z, \omega) = z$ and $F(\omega, z) = \omega$, that is, (z, ω) is a coupled fixed point of F.

6 Uniqueness

In this section we study the uniqueness of the coupled fixed point in two senses.

Theorem 11. *Under the hypotheses of Theorem 10, let (x,y) and (x',y') be two coupled fixed points of F such that there exists $(z,\omega)\in X^2$ verifying, at least, one of the following conditions hold:*

- $z\preceq x,\quad z\preceq x',\quad \omega\succeq y,\quad \omega\succeq y';$ (11)
- $z\preceq x,\quad z\succeq x',\quad \omega\succeq y,\quad \omega\preceq y';$ (12)
- $z\succeq x,\quad z\preceq x',\quad \omega\preceq y,\quad \omega\succeq y';$ (13)
- $z\succeq x,\quad z\succeq x',\quad \omega\preceq y,\quad \omega\preceq y'.$ (14)

Then $(x,y)=(x',y')$, that is, both are the same coupled fixed point.

Proof. Let (x,y) and (x',y') be arbitrary coupled fixed points of F. By hypothesis, there exists $(z,\omega)\in X^2$ such that, at least, one of the four conditions (11)-(14) holds. Let $\{z_n\}$ and $\{\omega_n\}$ be the sequences defined by

$$(z_0,\omega_0)=(z,\omega)\quad\text{and}$$
$$(z_{n+1},\omega_{n+1})=(F(x_n,y_n),F(y_n,x_n))\quad\text{for all } n\in\mathbb{N}.$$

We are going to show that $\{x_n\}$ converges to x and to x', and that $\{y_n\}$ converges to y and to y' (in such a case, $x=x'$ and $y=y'$, so F has a unique coupled fixed point). Let assume that (11) holds (the other cases are similar). Using (x,y), we claim that $\{z_n\}\to x$ and $\{\omega_n\}\to y$ (if we take (x',y'), we will deduce a similar conclusion). Indeed, as $z_0=z\preceq x$ and $\omega_0=\omega\succeq y$, then the mixed $\preceq$-monotone property of F proves that

$$z_1=F(z_0,\omega_0)\preceq F(x,\omega_0)\preceq F(x,y)=x\quad\text{and}$$
$$\omega_1=F(\omega_0,z_0)\succeq F(y,z_0)\succeq F(y,x)=y.$$

As $\preceq$ is transitive, then $z_1\preceq x$ and $\omega_1\succeq y$. By induction, it can be proved that $z_n\preceq x$ and $\omega_n\succeq y$ for all $n\in\mathbb{N}$. Applying the contractivity condition (7), it follows that, for all $n\in\mathbb{N}$,

$$\begin{aligned}\psi(d(z_{n+1},x))&=\psi(d(F(z_n,\omega_n),F(x,y)))\\&\le\psi(\max\{d(z_n,x),d(\omega_n,y)\})\\&\quad-\phi(\max\{d(z_n,x),d(\omega_n,y)\}).\end{aligned}$$

Similarly, as $y\preceq\omega_n$ and $x\succeq z_n$ for all $n\in\mathbb{N}$,

$$\begin{aligned}\psi(d(\omega_{n+1},y))&=\psi(d(F(y,x),F(\omega_n,z_n)))\\&\le\psi(\max\{d(y,\omega_n),d(x,z_n)\})\\&\quad-\phi(\max\{d(y,\omega_n),d(x,z_n)\})\\&=\psi(\max\{d(z_n,x),d(\omega_n,y)\})\\&\quad-\phi(\max\{d(z_n,x),d(\omega_n,y)\}).\end{aligned}$$

Joining the last two inequalities and taking into account that ψ is non-decreasing, for all $n\in\mathbb{N}$,

$$\begin{aligned}&\psi(\max\{d(z_{n+1},x),d(\omega_{n+1},y)\})\\&\quad=\max\{\psi(d(z_{n+1},x)),\psi(d(\omega_{n+1},y))\}\\&\quad\le\psi(\max\{d(z_n,x),d(\omega_n,y)\})\\&\qquad-\phi(\max\{d(z_n,x),d(\omega_n,y)\}).\end{aligned}$$

Lemma 3 guarantees that

$$\lim_{n\to\infty}\max\{d(z_n,x),d(\omega_n,y)\}=0,$$

that is, $\{z_n\}\to x$ and $\{\omega_n\}\to y$. The other cases are similar, so $\{z_n\}\to x'$ and $\{\omega_n\}\to y'$. As a consequence, $x=x'$ and $y=y'$.

Corollary 7. *Under the hypotheses of Theorem 10, also assume that the following condition holds:*

(U_1) *for all coupled fixed points (x,y) and (x',y') of F, there exists $(z,\omega)\in X^2$ verifying, at least, one of the conditions (11)-(14).*

Then F has a unique coupled fixed point.

Proof. If (x,y) and (x',y') are two arbitrary coupled fixed points of F, then Theorem 11 guarantees that $(x,y)=(x',y')$, so F has a unique coupled fixed point.

The previous result does not say how is the unique fixed point. The following one is more specific. Notice that we do not assume hypothesis (U).

Theorem 12. *Under the hypotheses of Theorem 10, let (x,y) be a coupled fixed point of F such that there exists $z\in X$ verifying, at least, one of the following conditions:*

- $y\preceq z\preceq x;$ (15)
- $x\preceq z\preceq y.$ (16)

Then $x=y$.

Proof. Let (x,y) be an arbitrary coupled fixed point of F. By hypothesis, there exists $z\in X$ such that, at least, one of the conditions (15)-(16) holds. Assume, for instance, that (15) holds (the other case is similar). Let consider the sequence

$$z_0=z\quad\text{and}\quad z_{n+1}=F(z_n,z_n)\quad\text{for all } n\in\mathbb{N}.$$

We claim that $\{z_n\}$ converges, at the same time, to x and to y. Indeed, as $y\preceq z=z_0\preceq x$, the mixed $\preceq$-monotone property guarantees that, for all $n\in\mathbb{N}$,

$$z_1=F(z_0,z_0)\preceq F(x,z_0)\preceq F(x,y)=x\quad\text{and}$$
$$z_1=F(z_0,z_0)\succeq F(y,z_0)\preceq F(y,x)=y.$$

As $\preceq$ is transitive, then $y \preceq z_1 \preceq x$. Reasoning by induction, we may prove that $y \preceq z_n \preceq x$ for all $n \in \mathbb{N}$. Using the contractivity condition (7) with $z_n \preceq x$ and $z_n \succeq y$, we have that

$$\begin{aligned}\psi(d(z_{n+1},x)) &= \psi(d(F(z_n,z_n),F(x,y))) \\ &\leq \psi(\max\{d(z_n,x),d(z_n,y)\}) \\ &\quad - \phi(\max\{d(z_n,x),d(z_n,y)\}).\end{aligned}$$

Similarly, as $y \preceq z_n$ and $x \succeq z_n$, then

$$\begin{aligned}\psi(d(z_{n+1},y)) &= \psi(d(y,z_{n+1})) \\ &= \psi(d(F(y,x),F(z_n,z_n))) \\ &\leq \psi(\max\{d(y,z_n),d(x,z_n)\}) \\ &\quad - \phi(\max\{d(y,z_n),d(x,z_n)\}).\end{aligned}$$

Joining the last two inequalities and taking into account that ψ is non-decreasing, for all $n \in \mathbb{N}$,

$$\begin{aligned}&\psi(\max\{d(z_{n+1},x),d(z_{n+1},y)\}) \\ &\quad = \max\{\psi(d(z_{n+1},x)),\psi(d(z_{n+1},y))\} \\ &\quad \leq \psi(\max\{d(z_n,x),d(z_n,y)\}) \\ &\qquad - \phi(\max\{d(z_n,x),d(z_n,y)\}).\end{aligned}$$

Lemma 3 guarantees that

$$\lim_{n\to\infty} \max\{d(z_{n+1},x),d(z_{n+1},y)\} = 0,$$

that is, $\{z_n\} \to x$ and $\{z_n\} \to y$. By the unicity of the limit of a convergent sequence in a metric space, then $x = y$.

Corollary 8. *Under the hypotheses of Theorem10, also assume that the following condition is fulfilled:*

(U_2) *for all coupled fixed point (x,y) of F, there exists $z \in X$ such that, at least, one of the conditions (15)-(16) holds.*

Then any coupled fixed point of F is of the form (x,x), for some $x \in X$.

Proof. If (x,y) is an arbitrary coupled fixed point of F, then $x = y$ by Theorem 12.

And, finally, we join assumptions (U_1) and (U_2) in the same result.

Corollary 9. *Under the hypotheses of Theorem10, also assume that conditions (U_1) and (U_2) hold. Then f has a unique coupled fixed point, which is of the form (x,x), for some $x \in X$.*

In particular, there exists a unique $x \in X$ such that $F(x,x) = x$.

Theorem 11 and Corollaries 8 and 9 can be particularized in several ways, as the consequences of Theorem 6 in Section 4. Finally, we only highlight the following statements: the first one is a particular version of the mentioned results using a partial order $\preccurlyeq$ on X (we leave to the reader this task), and the second one can be deduced using the preorder $\preceq_0$ given in (6).

Corollary 10. *Let (X,d) be a complete metric space and let $F : X \times X \to X$ be a given mapping. Suppose that there exist functions $\psi \in \mathscr{F}$ and $\phi \in \mathscr{F}'_{\mathrm{alt}}$ such that*

$$\begin{aligned}\psi(d(F(x,y),F(u,v))) &\leq \psi(\max\{d(x,u),d(y,v)\}) \\ &\quad - \phi(\max\{d(x,u),d(y,v)\})\end{aligned}$$

for all $x,y,u,v \in X$. Then F has a unique coupled fixed point, which is of the form (x,x) for some $x \in X$.

Now, we deduce the following consequences.

Theorem 13. *Gnana Bhaskar and Laksmikhantam's coupled fixed point theorems follows from Theorem 11 and Corollaries 8 and 9.*

As conclusion, we must highlight that Ran and Reuring's theorem and Nieto and Rodríguez-López's theorem are different faces of a same theorem, using a common condition.

Following the same techniques, it is possible to show existence and uniqueness theorems for Berinde and Borcut's tripled fixed points and Karapınar's quadrupled fixed points.

Competing interests

The author declares that there is no conflict of interests regarding the publication of this article.

Authors' contributions

The author completed the paper by himself. The author read and approved the final manuscript.

Acknowledgements

The author is grateful to the Department of Quantitative Methods for Economics and Business of the University of Granada (Spain) for its economic support. This manuscript has been partially supported by Junta de Andalucía by project FQM-268 of the Andalusian CICYE.

References

[1] R. Agarwal, E. Karapınar, A. Roldán, Fixed point theorems in quasi-metric spaces and applications to coupled/tripled fixed points on G^*-metric spaces, to appear in *Journal of Nonlinear and Convex Analysis*.

[2] Bhaskar, T.G., Lakshmikantham, V., Fixed Point Theory in partially ordered metric spaces and applications, Nonlinear Anal. 65 (2006) 1379–1393.

[3] V. Lakshmikantham, L. Ciric, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Anal. 70 (2009) 4341–4349.
[4] V. Berinde, Coupled fixed point theorems for ϕ-contractive mixed monotone mappings in partially ordered metric spaces, Nonlinear Anal. 75 (2012) 3218–3228.
[5] V. Berinde, M. Borcut, Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces, Nonlinear Anal. 74 (2011) 4889–4897.
[6] M. Berzig, E. Karapınar and A. Roldán, Discussion on generalized-$(\alpha\psi,\beta\varphi)$-contractive mappings via generalized altering distance function and related fixed point theorems, Abstr. Appl. Anal. 2014, Article ID 259768, 12 pages.
[7] M. Berzig, B. Samet, An extension of coupled fixed point's concept in higher dimension and applications, Comput. Math. Appl. 63 (2012) 1319–1334.
[8] M. Borcut, V. Berinde, Tripled coincidence theorems for contractive type mappings in partially ordered metric spaces, Appl. Math. Comput. 218 (2012), no:10, 5929–5936.
[9] D.W. Boyd, T.S.W. Wong, On nonlinear contractions, Proc. Amer. Math. Soc. 20 (1969) 458-464.
[10] H.-S. Ding, E. Karapınar, E. Meir Keeler Type Contractions in partially ordered *G*-metric space Fixed Point Theory and Applications, 2013:35, (2013).
[11] P.N. Dutta, B.S. Choudhury, A generalization of contraction principle in metric spaces, Fixed Point Theory Appl. 2008, Article ID 406368, 8 pages.
[12] J. Harjani, K. Sadarangani, Generalized contractions in partially ordered metric spaces and applications to ordinary differential equations. Nonlinear Anal. 72 (2010) 1188 1197.
[13] E. Karapınar, Quadruple fixed point theorems for weak ϕ-contractions, ISRN Mathematical Analysis, 2011, Article ID 989423, 16 pages (2011).
[14] E. Karapinar and R.P. Agarwal, Further fixed point results on G-metric spaces, Fixed Point Theory and Applications, 1 (2013) 2013:154
[15] E. Karapınar, N.V. Luong, Quadruple fixed point theorems for nonlinear contractions, Comput. Math. Appl., 64 (6) (2012), 1839–1848.
[16] E. Karapınar, A. Roldán, J. Martínez-Moreno and C. Roldán, Meir-Keeler type multidimensional fixed point theorems in partially ordered metric spaces, Abstract and Applied Analysis 2013, Article ID 406026.
[17] M.S. Khan, M. Swaleh, and S. Sessa. Fixed point theorems by altering distances between the points. Bull. Aust. Math. Soc. 30 (1) (1984) 1–9.
[18] J.J. Nieto, R. Rodríguez-López, Contractive mapping theorem in partially ordered sets and applications to ordinary differential equations, Order 22 (2005) 223–239.
[19] A.C.M. Ran, M.C.B. Reurings, A fixed point theorem in partially ordered sets and some applications to matrix equations, Proc. Amer. Math. Soc. 132 (2004) 1435–1443.
[20] A. Roldán, E. Karapınar, Some multidimensional fixed point theorems on partially preordered G^*-metric spaces under (ψ,ϕ)-contractivity conditions, Fixed Point Theory Appl. 2013, 2013:158.
[21] A. Roldán, J. Martínez-Moreno, C. Roldán, Multidimensional fixed point theorems in partially ordered complete metric spaces, Journal of Mathematical Analysis and Applications 396 (2) (2012) 536–545.
[22] A. Roldán, J. Martínez-Moreno, C. Roldán, and E. Karapınar, Multidimensional fixed point theorems in partially ordered complete partial metric spaces under (ψ,φ)-contractivity conditions, Abstract and Applied Analysis 2013, Article ID 634371.
[23] A. Roldán, E. Karapınar and P. Kumam, *G*-Metric spaces in any number of arguments and related fixed-point theorems, Fixed Point Theory and Applications 2014, 2014:13.
[24] A. Roldán, J. Martínez-Moreno, C. Roldán, Y.J. Cho, Multidimensional coincidence point results for compatible mappings in partially ordered fuzzy metric spaces, Fuzzy Sets Syst. **251** (2014) 71–82
[25] S.A. Al-Mezel, H.H. Alsulami, E. Karapınar and A. Roldán, Discussion on "Multidimensional coincidence points" via recent publications, Abstr. Appl. Anal. 2014, Article ID 287492, 13 pages.
[26] Choudhury, B.S., Kundu, A.: A coupled coincidence point result in partially ordered metric spaces for compatible mappings, Nonlinear Anal. *73* (2010) 2524–2531.
[27] Luong, N.V. and Thuan, N.X., Coupled points in ordered generalized metric spaces and application to integro-differential equations. Comput. Math. Appl. **62** (11) (2011) 4238–4248.
[28] Hung, N.M., Karapınar, E., Luong, N.V.: Coupled coincidence point theorem for *O*-compatible mappings via implicit relation, Abstr. Appl. Anal. Volume 2012 (2012), Article ID 796964, 14 pages.
[29] B. Schweizer, A. Sklar, Probabilistic Metric Spaces, Dover Publications, New York, 2005.
[30] S. Wang, Coincidence point theorems for *G*-isotone mappings in partially ordered metric spaces, Fixed Point Theory Appl. (2013), 2013:96.

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