Quantum Information Review
An International Journal

http://dx.doi.org/10.18576/qir/040201

Effects of Environment on Atom-Cavity, Atom-Atom and Cavity-Cavity Entanglement

*Biplab Ghosh**

Dept. of Physics, Vivekananda College for Women, Barisha, Kolkata-700008, India

Received: 23 Jan. 2016, Revised: 3 Feb. 2016, Accepted: 15 Feb. 2016
Published online: 1 Jul. 2016

Abstract: We investigate the various types of entanglement such as atom-cavity entanglement, atom-atom entanglement and cavity-cavity entanglement in an ideal cavity as well as their evolution in presence of cavity dissipation or environment. We quantify the various entanglement in ideal situation and they are plotted versus the Rabi angle gt. Next, we discuss the above cases in presence of cavity dissipation. Basically we present a comparative study of atom-cavity, atom-atom and cavity-cavity entanglement for ideal and realistic situation using two entanglement measures concurrence and the negativity or logarithmic negativity. We have seen sudden death of entanglement in finite time induced by environment for large value of cavity leakage parameter κ.

Keywords: Entanglement, Jaynes-Cummings Model, Cavity-QED, Concurrence, Negativity, Logarithmic negativity

1 Introduction

The esssence of nonlocality [1,2] follows from the property of inseparability of composite quantum systems. Consider two particles that once interacted but are remote from one another now and do not interact. Although they do not interact, they are still *entangled* if *their joint state cannot be written as a product of the states of individual subsystems*. Schrödinger [3] first coined the term 'entanglement' for the non-local correlation represented by the inseparable state. Such states are now called entangled states. Quantum entanglement is one of the essential ingredients in the current development of quantum information processing. Now entanglement is treated as a resource in quantum communication and computation protocols [4,5]. After Bell's work quantum entanglement became a subject of intensive study among those interested in the foundations of quantum theory. But more recently, entanglement has come to be viewed not just as a tool for exposing the weirdness of quantum mechanics, but as a potentially valuable resource. By exploiting entangled quantum states, we can perform tasks that are otherwise difficult or impossible i.e., typical resources required for cryptography, quantum teleportation,dense-coding [4] and controlled dense-coding [6] are entangled states. For example, in entanglement-assisted teleportation entangled pairs are used (one maximally entangled qubit pair is needed for every qubit teleported).

The arena of atom-photon interactions is a vast and potentially useful physical domain for implementing quantum information protocols. Entanglement has been widely observed in quantum optical systems such as cavity quantum electrodynamics. A number of experiments have been carried out. Several studies have been performed to quantify the entanglement that is obtained in atom-photon interactions in a cavity [7]-[14], which, from the view point of information processing, is considered an important aspect. Practical realization of various features of quantum entanglement are obtained in atom-photon interactions in optical and microwave cavities, using which controlled experiments can be performed with the present state-of-the-art technology. In this paper we perform the study of several facets of quantum entanglement generated in atom-photon interactions with the viewpoint of obtaining interesting and useful applications in real physical processes and devices.

The structure of the paper is as follows. In Sec. 2, we review briefly the the Jaynes-Cummings model. In Sec. 3, we investigate the dissipatives dynamics. In Sec. 4, we show the various types of entanglement such as atom-cavity entanglement, atom-atom entanglement and cavity-cavity entanglement in an ideal cavity as well as

* Corresponding author e-mail: quantumroshni@gmail.com

their evolution in presence cavity dissipation. A summary of our results and some concluding remarks are presented in Sec. 5.

2 The Jaynes-Cummings model

One of the most fundamental models in quantum mechanics presented in introductory text books is that of the two-level system and the harmonic oscillator. Combining these two into a bipartite system gives many interesting results using one of the most studied models, i.e., the Jaynes Cummings (JC) model [15]. The JC model is the simplest fully quantized model describing the interaction between a two-level atom and a quantized electromagnetic field. The model consists of a single two-level atom interacting with a single quantized electromagnetic cavity field (Figure 1).

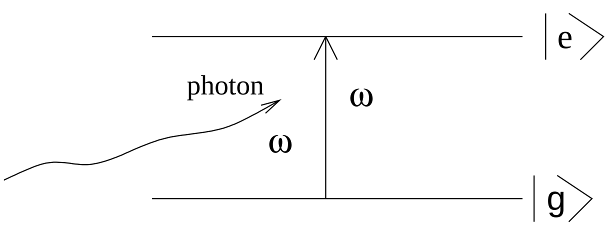

Fig. 1 A two-level atom-photon interaction.

The Jaynes-Cummings Hamiltonian is obtained by simply imposing the rotating wave approximation RWA [16]. In this approximation exact analytical solutions exist, and in spite of the simplicity of the JC model, the dynamics have turned out to be very rich and complex, describing well several physical phenomena. Among these, atom-field entanglement [17,18,19] is a very interesting subject of research. We have used the Jaynes-Cummings interaction to investigate atom-cavity, atom-atom and cavity-cavity entanglements. In this paper we will disscuss atom-cavity, atom-atom and cavity-cavity entanglement in detail without and with cavity field dissipation.

A two level atom is formally analogous to a spin-$1/2$ system with two possible states. Let us denote the upper level of the atom as $|e\rangle$ and the lower level as $|g\rangle$. Here we can write the step up and the step down operator as $\sigma^+ = |e\rangle\langle g|$ and $\sigma^- = |g\rangle\langle e|$, with the commutation relation

$$\begin{aligned}[\sigma^+,\sigma^-] &= |e\rangle\langle e| - |g\rangle\langle g| \\ &= \sigma_z. \end{aligned} \tag{1}$$

A quantum mechanical field can be represented as (for present purpose, we consider a single mode field)

$$E(t) = \frac{\mathscr{E}}{2}[ae^{-i\omega t} + a^\dagger e^{i\omega t}] \tag{2}$$

apart from a mode function which we omit here as it is not required for the present discussion. Here a and $a^\dagger$ are annihilation and creation operators, respectively, ω is the frequency of the field and $\mathscr{E}$ has the dimension of electric field. The graininess of the radiation field is represented by the photon number state $|n\rangle$, $n = 0,1,2,....$, such that $a|n\rangle = \sqrt{n}|n-1\rangle$ and $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$. It is an eigenstate of the number operator $\hat{n} = a^\dagger a, \hat{n}|n\rangle = n|n\rangle$. The field in Eq.(2) can be represented by a quantum mechanical state vector $|\psi\rangle$ which is a linear superposition of the number states $|n\rangle$, that is

$$|\psi\rangle = \sum_{n=0}^{\infty} c_n|n\rangle \tag{3}$$

where c_n is, in general, complex and gives the probabilty that the field has n photons by the relation

$$P_n = \langle n|\psi\rangle\langle\psi|n\rangle = |c_n|^2 \tag{4}$$

It is now a quantum statistical field and its average photon number is given by

$$< n >= \sum_{n=0}^{\infty} nP_n \tag{5}$$

with the intensity of the field $I \propto < n >$. The statistics brings in a quantum mechanical noise which is represented by the variance

$$V = \frac{< n^2 > - < n >}{< n^2 >}. \tag{6}$$

$V = 1$ is for coherent state field and $V < 1$ signifies a non-classical state. The parameters $< n >$ and V give a fair description of the quantum mechanical nature of the radiation field. The interaction between the two-level atom and the single mode field can be written in the dipole approximation as, $H_{int} = d.E/\hbar$. Here H_{int} is in frequency units, and d is the dipole moment of the atom which can be witten as $d = -\langle e|x|g\rangle$. Writing E in terms of operators in Eq.(2), and the dipole moment by spin operators in Eq.(1), the interacting atom-field system can be represented by the Hamiltonian

$$H = H_0 + H_{int}, \tag{7}$$

where the unperturbed Hamiltonian $H_0 = \frac{\Omega\sigma_z}{2} + (a^\dagger a + \frac{1}{2})\omega$ and $H_{int} = g(\sigma^+ + \sigma^-)(a + a^\dagger)$ and $g = -\frac{d\mathscr{E}}{\hbar}$ is the coupling constant. In a frame rotating at frequency $\omega[\sigma_z/2 + (a^\dagger a + 1)]$, the equation of motion defining the system is

$$i\frac{\partial}{\partial t}|\psi(t)\rangle_I = H_I|\psi(t)\rangle_I, \tag{8}$$

where the Hamiltonian H reduces to

$$H_I = g(\sigma^+ a + \sigma^- a^\dagger). \tag{9}$$

We deal with this interaction Hamiltonian H_I to investigate the atom-cavity, atom-atom and cavity-cavity entanglements in this paper.

3 Dissipative dynamics

Let us now investigate the dynamics of atom-photon interactions in the presence of cavity dissipation. Since the lifetime of a two-level Rydberg atom is usually much longer compared to the atom-cavity interaction time, we can safely neglect the atomic dissipation. The dynamics of the atom-photon interaction is governed by the evolution equation

$$\dot{\rho} = \dot{\rho}|_{\text{atom-field}} + \dot{\rho}|_{\text{field-reservoir}}, \tag{10}$$

where the strength of the couplings are given by the parameters κ (the cavity leakage constant) and g (the atom-field interaction constant). $\dot{\rho}|_{\text{atom-field}} = -i[H_I, \rho_{\text{atom-field}}]$. The reservoir-induced interactions can be effectively represented by the well-known master equations [20,21] using Born and Markoff approximations. For the reservoir coupling we have, after tracing over the reservoir variables,

$$\begin{aligned}\dot{\rho}|_{\text{field-reservoir}} &= -\kappa(1+<n>)(a^\dagger a\rho - 2a\rho a^\dagger + \rho a^\dagger a) \\ &\quad -\kappa <n> (aa^\dagger\rho - 2a^\dagger\rho a + \rho aa^\dagger),\end{aligned} \tag{11}$$

where $<n>$ is average thermal photons at the cavity temperature T. A derivation of this equation is given in the Appendix 1 on page 114. At temperature $T = 0K$ the average thermal photon number is zero, and hence one has [21]

$$\dot{\rho}|_{\text{field-reservoir}} = -\kappa(a^\dagger a\rho - 2a\rho a^\dagger + \rho a^\dagger a). \tag{12}$$

The total dynamical equation for atom-field density state ρ, is thus given by

$$\dot{\rho} = -i[H_I, \rho_{\text{atom-field}}] - \kappa(a^\dagger a\rho - 2a\rho a^\dagger + \rho a^\dagger a). \tag{13}$$

3.1 A model solution

In cavity-QED, one usually has $g \gg \kappa$. Hence, in most cases, it is sufficient to get a solution of Eq.(13) to the first order in κ. It is straightforward to express the damping equation for the density matrix elements in the dressed state basis.

$$\begin{aligned}\langle +,0|\dot{\rho}|+,0\rangle &= -\kappa\langle +,0|\rho|+,0\rangle - \frac{\kappa}{2}\langle -,0|\rho|+,0\rangle \\ &\quad -\frac{\kappa}{2}\langle +,0|\rho|-,0\rangle.\end{aligned} \tag{14}$$

$$\begin{aligned}\langle -,0|\dot{\rho}|-,0\rangle &= -\kappa\langle -,0|\rho|-,0\rangle - \frac{\kappa}{2}\langle +,0|\rho|-,0\rangle \\ &\quad -\frac{\kappa}{2}\langle -,0|\rho|+,0\rangle.\end{aligned} \tag{15}$$

$$\begin{aligned}\langle +,0|\dot{\rho}|-,0\rangle &= -2ig\langle +,0|\dot{\rho}|-,0\rangle - \kappa\langle +,0|\rho|-,0\rangle \\ &\quad -\frac{\kappa}{2}\langle +,0|\rho|+,0\rangle - \frac{\kappa}{2}\langle -,0|\rho|-,0\rangle.\end{aligned} \tag{16}$$

We note that the terms $\langle +,0|\rho|+,0\rangle$ and $\langle -,0|\rho|-,0\rangle$ oscillate at zero frequency (or donot oscillate), whereas the terms $\langle +,0|\dot{\rho}|-,0\rangle$ oscillate at frequency g. The strength of the coupling of these terms are of the order of κ. Hence, for $g \gg \kappa$, it is reasonable to assume that they decouple. In other words, we can neglect their coupling. In the literature, such an approximation is called the "secular approximation". Under this approximation, the equations of motion reduce to

$$\langle +,0|\dot{\rho}|+,0\rangle = -\kappa\langle +,0|\rho|+,0\rangle. \tag{17}$$

$$\langle -,0|\dot{\rho}|-,0\rangle = -\kappa\langle -,0|\rho|-,0\rangle. \tag{18}$$

$$\langle +,0|\dot{\rho}|-,0\rangle = -2ig\langle +,0|\dot{\rho}|-,0\rangle - \kappa\langle +,0|\rho|-,0\rangle. \tag{19}$$

The obvious solutions of Eq.(17), Eq.(18) and Eq.(19) are

$$\langle +,0|\rho|+,0\rangle_t = e^{-\kappa t}\langle +,0|\rho|+,0\rangle_{t=0}, \tag{20}$$

$$\langle -,0|\rho|-,0\rangle_t = e^{-\kappa t}\langle -,0|\rho|-,0\rangle_{t=0}, \tag{21}$$

$$\langle +,0|\rho|-,0\rangle_t = e^{-2igt}e^{-\kappa t}\langle +,0|\rho|-,0\rangle_{t=0}. \tag{22}$$

We also work under a further approximation (that is justified when the cavity is close to $0K$) that the probability of getting two or more photons inside the cavities is zero, or in other words, the cavity always remains in the two-level state comprising of $|0>$ and $|1>$. For example, the initial state $|e,0\rangle$ corresponds to the boundary condition

$$\langle +,0|\rho|+,0\rangle_{t=0} = \langle -,0|\rho|-,0\rangle_{t=0} = \frac{1}{2}, \tag{23}$$

and

$$\langle +,0|\rho|-,0\rangle_{t=0} = -\frac{1}{2}. \tag{24}$$

Therefore

$$\begin{aligned}\langle e,0|\rho|e,0\rangle_t &= \frac{1}{2}[\langle +,0|\rho|+,0\rangle_t + \langle -,0|\rho|-,0\rangle_t \\ &\quad - \langle +,0|\rho|-,0\rangle_t - \langle -,0|\rho|+,0\rangle_t] \\ &= \frac{1}{2}e^{-\kappa t}[1+\cos 2gt] \\ &= e^{-\kappa t}\cos^2 gt,\end{aligned} \tag{25}$$

$$\begin{aligned}\langle g,1|\rho|g,1\rangle_t &= \frac{1}{2}[\langle +,0|\rho|+,0\rangle_t + \langle -,0|\rho|-,0\rangle_t \\ &\quad + \langle +,0|\rho|-,0\rangle_t + \langle -,0|\rho|+,0\rangle_t] \\ &= \frac{1}{2}e^{-\kappa t}[1-\cos 2gt] \\ &= e^{-\kappa t}\sin^2 gt,\end{aligned} \tag{26}$$

and

$$\begin{aligned}\langle e,0|\rho|g,1\rangle_t &= \frac{1}{2}[\langle +,0|\rho|+,0\rangle_t + \langle +,0|\rho|-,0\rangle_t \\ &\quad - \langle -,0|\rho|+,0\rangle_t - \langle -,0|\rho|-,0\rangle_t] \\ &= \frac{i}{2}e^{-\kappa t}[\sin 2gt] \\ &= ie^{-\kappa t}\sin gt\cos gt,\end{aligned} \tag{27}$$

$$\langle g,1|\rho|e,0\rangle_t = -ie^{-\kappa t}\sin gt\cos gt. \tag{28}$$

The initial state $|g,1\rangle$ corresponds to the boundary condition

$$\langle +,0|\rho|+,0\rangle_{t=0} = \langle -,0|\rho|-,0\rangle_{t=0} = \frac{1}{2}, \tag{29}$$

and

$$\langle +,0|\rho|-,0\rangle_{t=0} = \frac{1}{2}. \tag{30}$$

Therefore

$$\begin{aligned}\langle e,0|\rho|e,0\rangle_t &= \frac{1}{2}[\langle +,0|\rho|+,0\rangle_t + \langle -,0|\rho|-,0\rangle_t \\ &- \langle +,0|\rho|-,0\rangle_t - \langle -,0|\rho|+,0\rangle_t] \\ &= \frac{1}{2}e^{-\kappa t}[1-\cos 2gt] \\ &= e^{-\kappa t}\sin^2 gt,\end{aligned} \tag{31}$$

$$\begin{aligned}\langle g,1|\rho|g,1\rangle_t &= \frac{1}{2}[\langle +,0|\rho|+,0\rangle_t + \langle -,0|\rho|-,0\rangle_t \\ &+ \langle +,0|\rho|-,0\rangle_t + \langle -,0|\rho|+,0\rangle_t] \\ &= \frac{1}{2}e^{-\kappa t}[1+\cos 2gt] \\ &= e^{-\kappa t}\cos^2 gt,\end{aligned} \tag{32}$$

and

$$\begin{aligned}\langle e,0|\rho|g,1\rangle_t &= \frac{1}{2}[\langle +,0|\rho|+,0\rangle_t + \langle +,0|\rho|-,0\rangle_t \\ &- \langle -,0|\rho|+,0\rangle_t - \langle -,0|\rho|-,0\rangle_t] \\ &= -\frac{i}{2}e^{-\kappa t}[\sin 2gt] \\ &= -ie^{-\kappa t}\sin gt\cos gt,\end{aligned} \tag{33}$$

$$\langle g,1|\rho|e,0\rangle_t = ie^{-\kappa t}\sin gt\cos gt. \tag{34}$$

The above method provides a typical way of solving cavity-QED coupled equations with dissipation. We use them here to study the effect of cavity dissipation on entanglement.

4 Various types of entanglement and the effect of cavity dissipation on them

In this section we investigate the various types of entanglement such as atom-cavity entanglement, atom-atom entanglement and cavity-cavity entanglement in an ideal cavity as well as their evolution in presence cavity dissipation. We quantify the entanglement either with the entanglement measure ‘concurrence’ or ‘entanglement of formation’ [22,23,24], through we know that for pure state the von Neumann entropy and ‘entanglement of formation’ are the same. For a comparison we will also consider the negativity or logarithmic negativity [25,26] to quantify the entanglement. The negativity and logarithmic negativity are defined as $N(\rho) = (||\rho^{T_A}||_1 - 1)/2$ and $E_N(\rho) = log_2||\rho^{T_A}||_1$ respectively, where ρ^{T_A} is the partial transpose of the bipartite mixed state ρ and $||\rho^{T_A}||_1$ is is the trace norm of ρ^{T_A}. The trace norm of any hermitian operator A is $||A_1|| = tr\sqrt{A^\dagger A}$ which is equal to the sum of the absolute values of the eigenvalues of A, when A is hermitian.

4.1 Atom-cavity entanglement

Let us first consider a two-level atom A_1 prepared in the excited state $|e\rangle$ passing through an empty cavity C_1 (Figure 6). The initial joint state of atom-cavity bipartite system is

$$|\Psi_{A_1C_1}(t=0)\rangle = |e\rangle\otimes|0\rangle. \tag{35}$$

The atom-field joint state evolves under the Jaynes-Cummings interaction to

$$|\Psi_{A_1C_1}(t)\rangle = \cos gt|e,0\rangle + \sin gt|g,1\rangle, \tag{36}$$

Therefore, the density state can be written as

$$\begin{aligned}\rho_{A_1C_1}(t) &= |\Psi_{A_1C_1}(t)\rangle\langle\Psi_{A_1C_1}(t)| \\ &= \cos^2 gt|e,0\rangle\langle e,0| + \cos gt\sin gt|e,0\rangle\langle g,1| \\ &+ \cos gt\sin gt|g,1\rangle\langle e,0| + \sin^2 gt|g,1\rangle\langle g,1|\end{aligned} \tag{37}$$

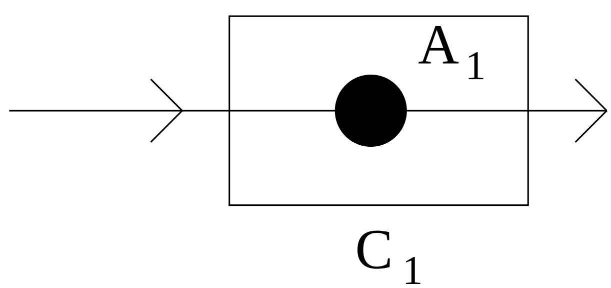

Fig. 2 A two-level atom prepared in the excited state is traversing through an epmty cavity.

The corresponding density matrix $\rho_{A_1C_1}(t)$ can be written as

$$\rho = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \cos^2 gt & \cos gt\sin gt & 0 \\ 0 & \cos gt\sin gt & \sin^2 gt & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{38}$$

in the basis $|e,1>, |e,0>, |g,1>$ and $|g,0>$ states. The concurrence C of $\rho_{A_1C_1}(t)$ is $2|\cos gt\sin gt|$. C is maximum $(=1)$ for Rabi angle $gt = (2n+1)\pi/4$. So for an interaction time $gt = (2n+1)\pi/4$, $|\Psi_{A_1C_1}(t)\rangle$ becomes maximally entangled and for an interaction time $gt = n\pi/2$, $|\Psi_{A_1C_1}(t)\rangle$ becomes disentangled.

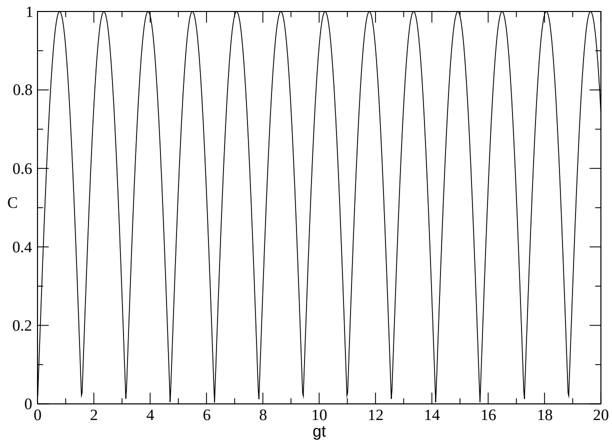

Fig. 3 Atom-cavity entanglement i.e., concurrence is plotted vs gt

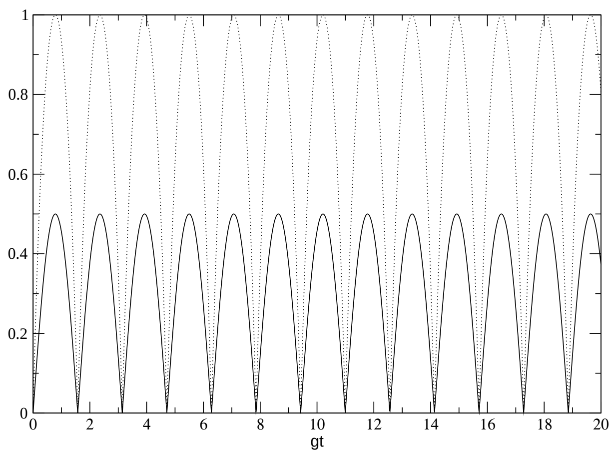

Fig. 4 $N_(\rho)$ (solid line) and $E_N(\rho)$(dotted line) of atom-cavity are plotted vs gt respectively.

In Figure 3 the concurrence C between the atom and the cavity is plotted versus the Rabi angle gt.

In Figure 4 the negativity and logarithmic negativity between the atom and the cavity are plotted versus the Rabi angle gt. We notice that for separable state $N_(\rho)$ becomes zero [25].

Next, we discuss the above case in presence of cavity dissipation. At temperature $T=0K$, the average thermal photon number is zero, and one has (see, for instance, Ref. [21])

$$\dot{\rho}|_{\text{field-reservoir}} = -\kappa(a^\dagger a\rho - 2a\rho a^\dagger + \rho a^\dagger a), \tag{39}$$

as in Eq.(12). When $g \gg \kappa$, it is possible to make the secular approximation [27] (discussed in section 3) to get the density elements of $\rho_{A_1C_1}(t)$. We also work under a further approximation (which is justified when the cavity is close to $0K$) that the probability of getting two or more photons inside the cavity is zero. The method of solving the dissipation equation has been outlined in section 3.1. The joint density state of atom and cavity is then obtained as

$$\begin{aligned}\rho_{A_1C_1}(t) = & (e^{-\kappa t}\cos^2 gt|e,0\rangle\langle e,0| + ie^{-\kappa t}\cos gt\sin gt|e,0\rangle\langle g,1| \\ & -ie^{-\kappa t}\cos gt\sin gt|g,1\rangle\langle e,0| + e^{-\kappa t}\sin^2 gt|g,1\rangle\langle g,1|),\end{aligned} \tag{40}$$

where κ is leakage constant for cavity C_1. The corresponding density matrix $\rho_{A_1C_1}(t)$ is given by

$$\rho = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & e^{-\kappa t}\cos^2 gt & ie^{-\kappa t}\cos gt\sin gt & 0 \\ 0 & -ie^{-\kappa t}\cos gt\sin gt & e^{-\kappa t}\sin^2 gt & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{41}$$

in the basis of $|e,1>, |e,0>, |g,1>$ and $|g,0>$ states.

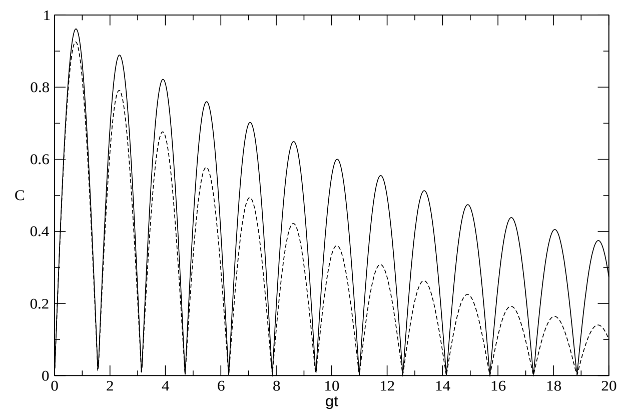

Fig. 5 Atom-cavity entanglement i.e., concurrence is plotted vs gt for (i) $\kappa/g = 0.05$ (solid line), (ii) $\kappa/g = 0.1$ (dashed line).

The concurrence C of $\rho_{A_1C_1}(t)$ is $|2e^{-\kappa t}\cos gt\sin gt|$. It is clear from the expression of the concurrence that it decreases with incease of κ and $C=0$ for $\kappa=\infty$. Two-qubit entanglement may terminate abruptly in a finite time [28], a phenomenon termed entanglement sudden death (ESD). In Figure 5 the concurrence C between the atom and the cavity is plotted versus the Rabi angle gt for different values of the cavity leakage constant κ/g. We see clearly the effect of dissipation on entanglement which reduces as we increase the cavity leakage constant κ. This shows that disipation reduces the atom-cavity entanglement and, ultimately it is destroyed at a later time. If we send an atom prepared in the ground state $|g\rangle$ through the one photon cavity, the initial joint atom-cavity state will be

$$|\Psi(t=0)\rangle = |g\rangle \otimes |1\rangle. \tag{42}$$

The time evolved state is

$$|\Psi(t)\rangle = \cos gt|g,1\rangle - \sin gt|e,0\rangle. \tag{43}$$

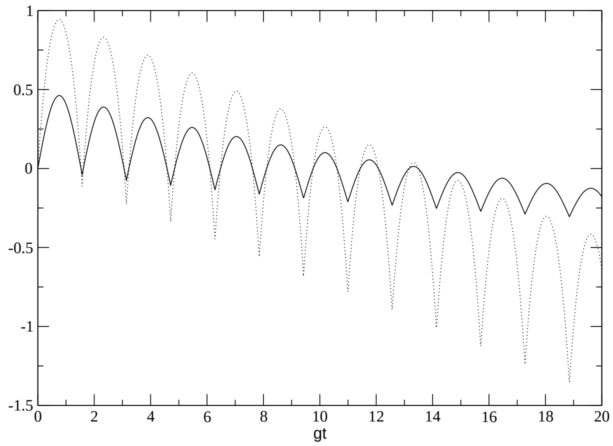

Fig. 6 Atom-cavity negativity (solid line) and logarithmic negativity (dotted line) are plotted vs gt for $\kappa/g = 0.05$

In this case the result for entanglement is similar to the case for the state

$$|\Psi(t)\rangle = \cos gt|e,0\rangle + \sin gt|g,1\rangle, \tag{44}$$

that we have considered earlier, both with and without dissipation.

In Figure 6 Atom-cavity negativity and logarithmic negativity are plotted vs gt for cavity leakage constant $\kappa/g = 0.05$. We notice that degree of magnetude has been reduced for both cases.

4.2 Atom-atom entanglement

We consider a system where two two-level atoms, the first prepared in the excited state and the second prepared in the ground state, are sent into a cavity in the vacuum state one after the other (see Figure 5). The flight times of both the atoms through the cavity are assumed to be the same.

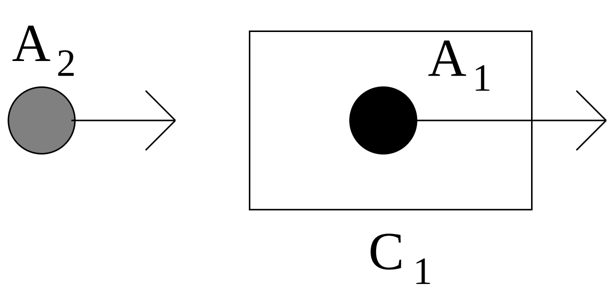

Fig. 7 Two two-level atoms, first prepared in the excited state and second prepared in the ground state, traverses an empty cavity one after the other.

Let us first consider the passage of the first atom, initially in the excited state $|e>$, through the cavity. The initial joint atom-field state is given by

$$|\Psi(t=0)\rangle_{A_1C_1} = |e\rangle \otimes |0\rangle. \tag{45}$$

The atom-field state evolves with the interaction given by Eqs.(36) to

$$|\Psi(t)_{A_1C_1}\rangle = \cos gt|e,0\rangle + \sin gt|g,1\rangle, \tag{46}$$

The next atom prepared in $|g>$ which enters the cavity interacts with this "changed" field and thus a correlation develops between the two atoms via the cavity field. The joint tripartite state of the two atoms and the field is given by

$$\begin{aligned} |\Psi(t)\rangle_{A_1A_2C_1} &= \cos gt|e_1,g_2,0\rangle + \cos gt \sin gt|g_1,g_2,1\rangle \\ &\quad - \sin^2 gt|g_1,e_2,0\rangle \end{aligned} \tag{47}$$

The corresponding atom-atom-field pure density state is

$$\begin{aligned} &\rho(t)_{A_1A_2C_1} = |\Psi(t)\rangle_{A_1A_2C_1\ A_1A_2C_1}\langle\Psi(t)| \\ &= \cos^2 gt|e_1,g_2,0\rangle\langle e_1,g_2,0| + \cos^2 gt\sin^2 gt|g_1,g_2,1\rangle\langle g_1,g_2,1| \\ &+ \sin^4 gt|g_1,e_2,0\rangle\langle g_1,e_2,0| + \cos^2 gt\sin gt|e_1,g_2,0\rangle\langle g_1,g_2,1| \\ &+ \cos^2 gt\sin gt|g_1,g_2,1\rangle\langle e_1,g_2,0| - \cos gt\sin^2 gt|e_1,g_2,0\rangle\langle g_1,e_2,0| \\ &- \cos gt\sin^2 gt|g_1,e_2,0\rangle\langle e_1,g_2,0| - \cos gt\sin^3 gt|g_1,g_2,1\rangle\langle g_1,e_2,0| \\ &- \cos gt\sin^3 gt|g_1,e_2,0\rangle\langle g_1,g_2,1|. \end{aligned} \tag{48}$$

The reduced density state of the pair A_1A_2 is obtained by tracing out the field variables, and is given by

$$\begin{aligned} &\rho(t)_{A_1A_2} = \mathrm{Tr}_{C_1}(\rho_{A_1A_2C_1}) \\ &= \cos^2 gt|e_1,g_2\rangle\langle e_1,g_2| + \cos^2 gt\sin^2 gt|g_1,g_2\rangle\langle g_1,g_2| \\ &+ \sin^4 gt|g_1,e_2\rangle\langle g_1,e_2| - \cos gt\sin^2 gt|g_1,e_2\rangle\langle e_1,g_2| \\ &- \cos gt\sin^2 gt|e_1,g_2\rangle\langle g_1,e_2| \end{aligned} \tag{49}$$

The corresponding density matrix $\rho_{A_1A_2}(t)$ is given by

$$\rho = \begin{pmatrix} \cos^2 gt\sin^2 gt & 0 & 0 & 0 \\ 0 & \sin^4 gt & -\cos gt\sin^2 gt & 0 \\ 0 & -\cos gt\sin^2 gt & \cos^2 gt & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{50}$$

in the basis $|g_1,g_2>, |g_1,e_2>, |e_1,g_2>$ and $|e_1,e_2>$ states.

We compute the concurrence for $\rho(t)_{A_1A_2}$ that is

$$C(\rho(t)_{A_1A_2}) = |2\cos gt\sin^2 gt|. \tag{51}$$

The concurrence between the two atoms is plotted versus the Rabi angle gt in Figure 8.

In Figure 9 the negativity and logarithmic negativity between the two atoms are plotted versus the Rabi angle gt.

We now investigate the above study in presence of the cavity dissipation. Like in the previous section, in the presence of cavity dissipation the evolved state of the system $A_1A_2C_1$ is a mixed state and is obtained with the

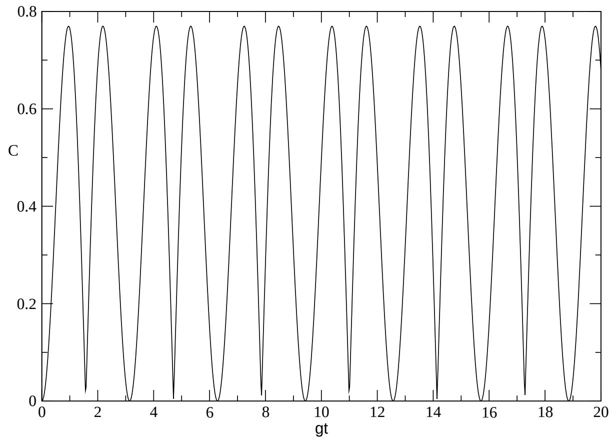

Fig. 8 Atom-atom entanglement i.e., concurrence is plotted vs gt

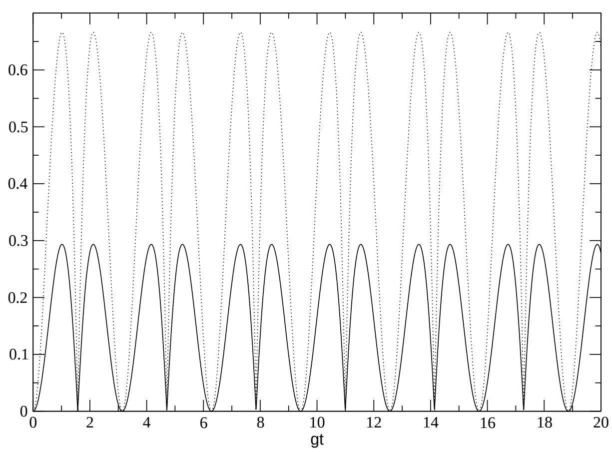

Fig. 9 $N_(\rho)$ (solid line) and $E_N(\rho)$(dotted line) of atom-atom are plotted vs gt respectively.

above approximations (see section 3.1). The reduced density state of the pair A_1A_2 is

$$\begin{aligned}\rho(t)_{A_1A_2} &= \mathrm{Tr}_{C_1}(\rho_{A_1A_2C_1}) \\ &= \gamma_1|e_1g_2\rangle\langle e_1g_2| \\ &+ \gamma_2|g_1g_2\rangle\langle g_1g_2| \\ &+ \gamma_3|g_1e_2\rangle\langle g_1e_2| \\ &- \gamma_4|e_1g_2\rangle\langle g_1e_2| \\ &- \gamma_4|g_1e_2\rangle\langle e_1g_2|, \end{aligned} \tag{52}$$

where the γ_i are given by

$$\gamma_1 = (1-\sin^2 gte^{-\kappa t}),$$
$$\gamma_2 = \cos^2 gt \sin^2 gte^{-2\kappa t},$$
$$\gamma_3 = \sin^4 gte^{-2\kappa t},$$

$$\gamma_4 = \left(\sin gte^{-\kappa t/2} - \frac{\kappa}{2g}\cos gte^{-\kappa t/2} + \frac{\kappa}{2g}\right)\cos gt \sin gte^{-\kappa t},$$

κ is the leakage constant of cavity C_1. The corresponding density matrix $\rho_{A_1A_2}(t)$ is

$$\rho = \begin{pmatrix} \gamma_2 & 0 & 0 & 0 \\ 0 & \gamma_3 & -\gamma_4 & 0 \\ 0 & -\gamma_4 & \gamma_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{53}$$

in the basis of $|g_1,g_2>, |g_1,e_2>, |e_1,g_2>$ and $|e_1,e_2>$ states.

We compute the concurrence for $\rho(t)_{A_1A_2}$, i.e.,

$$C(\rho(t)_{A_1A_2}) = |2\sin^2 gte^{-\kappa t}\sqrt{(1-\sin^2 gte^{-\kappa t})}| \tag{54}$$

ESD [28] is realized for $\kappa = \infty$.

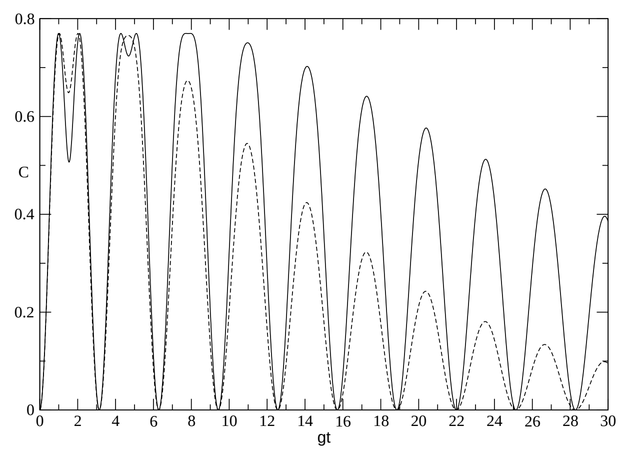

Fig. 10 Atom-atom entanglement i.e., concurrence is plotted vs gt for (i) $\kappa/g = 0.05$ (solid line) (ii) $\kappa/g = 0.1$ (dashed line).

The concurrence between the two atoms is plotted versus the Rabi angle gt in Figure 10 for different values of the cavity dissipation parameter κ.

The entanglement reduces as we increase κ. The effect of κ gets more and more pronouced as time increases. In Figure 11 Atom-atom negativity and logarithmic negativity are plotted vs gt for cavity leakage constant $\kappa/g = 0.05$. A similar trend has been observed.

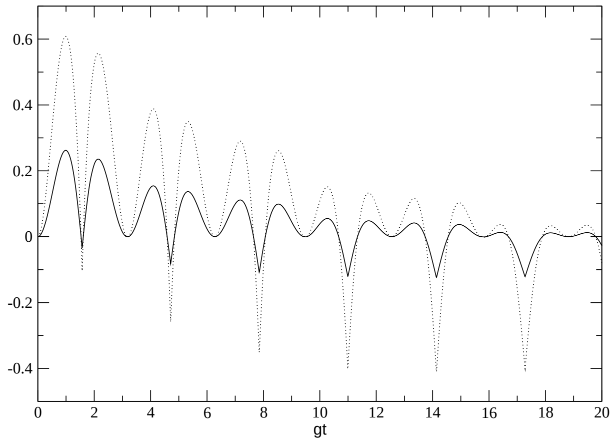

Fig. 11 Atom-atom negativity (solid line) and logarithmic negativity (dotted line) are plotted vs gt for $\kappa/g = 0.05$

4.3 Cavity-cavity entanglement

Here we consider two initially separated empty cavities C_1, C_2 and a two-level atom prepared in the excited state passing through C_1 and C_2 such that the times of flight of the atom through the two cavities are the same (see Figure 12).

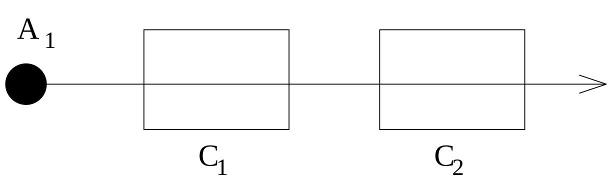

Fig. 12 A two-level atom prepared in the excited state is traversing through two separated cavities one after another.

The initial joint state of the atom and the two cavities is

$$|\Psi(t)\rangle_{A_1C_1C_2} = |e_1\rangle \otimes |0_1\rangle \otimes |0_2\rangle. \quad (55)$$

The joint state of the two cavities and the atom undergoing the Jaynes-Cummings interaction at a later time is

$$|\Psi(t)\rangle_{A_1C_1C_2} = \cos^2 gt|e_1, 0_2, 0_2\rangle + \cos gt \sin gt|g_1, 0_1, 1_2\rangle - \sin gt|g_1, 1_1, 0_2\rangle. \quad (56)$$

The corresponding cavity-cavity-atom tripartite pure density state is

$$\begin{aligned}
&\rho(t)_{A_1C_1C_2} = |\Psi(t)\rangle_{A_1C_1C_2}\ {}_{A_1C_1C_2}\langle\Psi(t)| \\
&= \cos^4 gt|e_1, 0_1, 0_2\rangle\langle e_1, 0_1, 0_2| + \cos^2 gt \sin^2 gt|g_1, 0_1, 1_2\rangle\langle g_1, 0_1, 1_2| \\
&+ \sin^2 gt|g_1, 1_1, 0_2\rangle\langle g_1, 1_1, 0_2| + \cos^3 gt \sin gt|e_1, 0_1, 0_2\rangle\langle g_1, 0_1, 1_2| \\
&+ \cos^2 gt \sin gt|e_1, 0_1, 0_2\rangle\langle g_1, 1_1, 0_2| + \cos^3 gt \sin gt|g_1, 0_1, 1_2\rangle\langle e_1, 0_1, 0_2| \\
&+ \cos gt \sin^2 gt|g_1, 0_1, 1_2\rangle\langle g_1, 1_1, 0_2| + \cos^2 gt \sin gt|g_1, 1_1, 0_2\rangle\langle e_1, 0_1, 0_2| \\
&+ \cos gt \sin^2 gt|g_1, 1_1, 0_2\rangle\langle g_1, 0_1, 1_2|. \qquad (57)
\end{aligned}$$

The reduced density state of the pair C_1C_2 is

$$\begin{aligned}
&\rho(t)_{C_1C_2} = \mathrm{Tr}_{A_1}(\rho_{A_1C_1C_2}) \\
&= \cos^4 gt|0_1, 0_2\rangle\langle 0_1, 0_2| + \cos^2 gt \sin^2 gt|0_1, 1_2\rangle\langle 0_1, 1_2| \\
&+ \sin^2 gt|1_1, 0_2\rangle\langle 1_1, 0_2| + \cos gt \sin^2 gt|1_1, 0_2\rangle\langle 0_1, 1_2| \\
&+ \cos gt \sin^2 gt|0_1, 1_2\rangle\langle 1_1, 0_2|. \qquad (58)
\end{aligned}$$

The corresponding density matrix $\rho_{C_1C_2}(t)$ is given by

$$\rho = \begin{pmatrix} \cos^4 gt & 0 & 0 & 0 \\ 0 & \sin^2 gt \cos^2 gt & \cos gt \sin^2 gt & 0 \\ 0 & \cos gt \sin^2 gt & \sin^2 gt & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (59)$$

in the basis of $|0_1, 0_2>, |0_1, 1_2>, |1_1, 0_2>$ and $|1_1, 1_2>$ states. We find that the concurrence for $\rho(t)_{C_1C_2}$ has the form

$$C(\rho(t)_{C_1C_2}) = |2\cos gt \sin^2 gt|. \quad (60)$$

The concurrences of the two atoms (see earlier section) and the two cavities are similar functions of the Rabi angle.

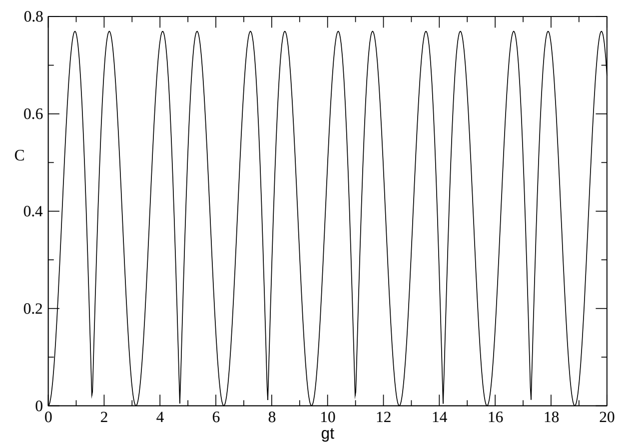

Fig. 13 Cavity-cavity entanglement i.e., concurrence is plotted vs gt

The concurrence between two cavities is plotted versus the Rabi angle gt in Figure 13.
In Figure 14 the negativity and logarithmic negativity between the two cavities are plotted versus the Rabi angle gt.

The time evolution of the reduced density state of two cavities in presence of dissipation is, following the method outlined earlier,

$$\begin{aligned}
&\rho(t)_{C_1C_2} = \mathrm{Tr}_{A_1}(\rho_{A_1C_1C_2}) \\
&= e^{-2\kappa t}\cos^4 gt|0_1, 0_2\rangle\langle 0_1, 0_2| + e^{-2\kappa t}\cos^2 gt \sin^2 gt|0_1, 1_2\rangle\langle 0_1, 1_2| \\
&+ (1 - e^{-\kappa t}\cos^2 gt)|1_1, 0_2\rangle\langle 1_1, 0_2| + e^{-\frac{3\kappa t}{2}}\cos gt \sin^2 gt|1_1, 0_2\rangle\langle 0_1, 1_2| \\
&+ e^{-\frac{3\kappa t}{2}}\cos gt \sin^2 gt|0_1, 1_2\rangle\langle 1_1, 0_2|. \qquad (61)
\end{aligned}$$

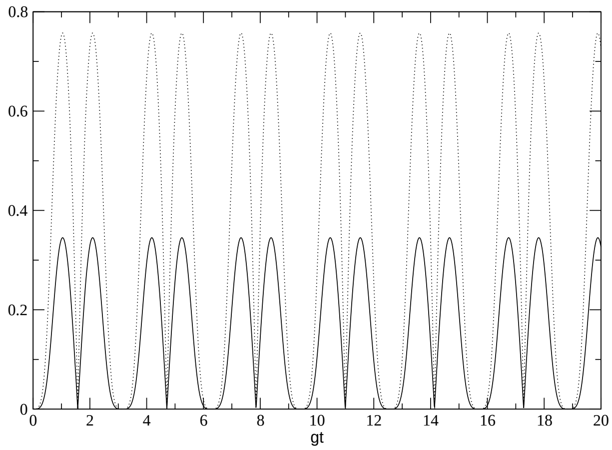

Fig. 14 $N_(\rho)$ (solid line) and $E_N(\rho)$(dotted line) of atom-atom are plotted vs gt respectively.

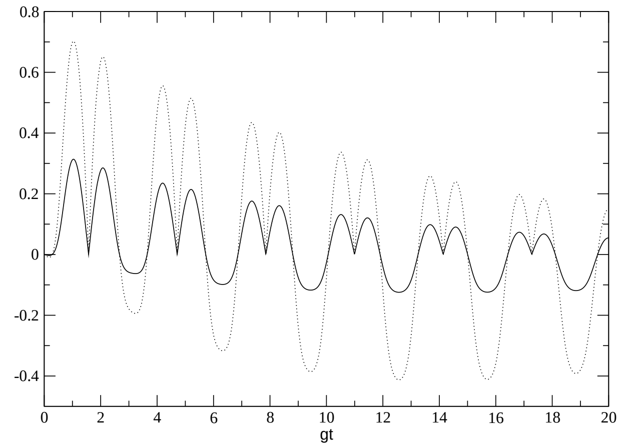

Fig. 16 Cavity-cavity negativity (solid line) and logarithmic negativity (dotted line) are plotted vs gt for $\kappa/g = 0.05$

The corresponding density matrix $\rho_{A_1A_2}(t)$ is given by

$$\rho = \begin{pmatrix} e^{-2\kappa t}\cos^4 gt & 0 & 0 & 0 \\ 0 & e^{-2\kappa t}\sin^2 gt\cos^2 gt & e^{-\frac{3\kappa t}{2}}\cos gt\sin^2 gt & 0 \\ 0 & e^{-\frac{3\kappa t}{2}}\cos gt\sin^2 gt & (1-e^{-\kappa t}\cos^2 gt) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (62)$$

in the basis of $|0_1,0_2>, |0_1,1_2>, |1_1,0_2>$ and $|1_1,1_2>$ states. We compute the concurrence for $\rho(t)_{C_1C_2}$, that is given by

$$C(\rho(t)_{C_1C_2}) = |2\cos gt\sin^2 gte^{-\frac{3\kappa t}{2}}|. \quad (63)$$

Here also we can realize ESD [28] for $\kappa = \infty$.

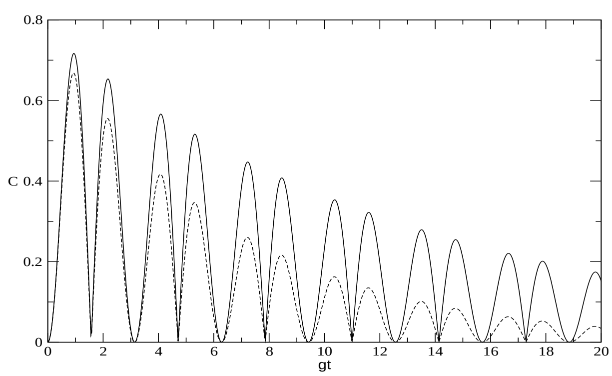

Fig. 15 Cavity-cavity entanglement i.e., concurrence is plotted vs gt for (i) $\kappa/g = 0.05$ (solid line) (ii) $\kappa/g = 0.1$ (dashed line).

The concurrence between two cavities is plotted versus the Rabi angle gt in Figure 15 for different values of the cavity dissipation parameter κ. The effect of κ gradually reduces the entanglement as it evolves in time. In Figure 15 cavity-cavity negativity and logarithmic negativity are plotted vs gt for cavity leakage constant $\kappa/g = 0.05$. A similar trend has been observed here also.

5 Summary

In this paper we have discussed the Jaynes-Cummings model which consists of a two-level atom interacting with a single mode cavity field. This is an exactly solvable model. We have considered resonant interaction between the atom and the cavity field. We saw that the Jaynes-Cummings interaction Hamiltonian takes a simple form in the rotating wave approximation. We then discussed the time evolution of atom-field states evolving under the Jaynes-Cummings interaction. Next we discussed dissipative dynamics of cavity field. Here we considered only cavity dissipation as the lifetime of two-level atoms (we are considering two-level Rydberg atoms) is much longer than the atom-field interaction time. Basically, the dynamical equation of the atom-field density state evolves under (i) the simple atom-field interaction, and (ii) the field-reservoir interaction. We solved the equations of motion under the secular approximation by using the fact that $g \gg \kappa$.

We investigated the various types of entanglement which origines from the Jaynes-Cummings interaction both in ideal and dissipative systems. We showed how the atom-cavity, atom-atom, and cavity-cavity entanglement can be generated in atom-photon interactions. We studied quantitatively the atom-cavity, atom-atom, and cavity-cavity entanglement as functions of the Rabi angle gt in the ideal situation and also in presence of cavity dissipation. We have used the entanglement measures concurrence [22,23] and logarithmic negativity (also negativity) [26, 25] to quantify the entanglement. We observed that the cavity dissipation kills the entanglement in all cases as we increase the atom-field interaction time. And also ESD [28] in a finite time for $\kappa = \infty$ has been realized from the expression of concurrence. In this paper we have set up the framework of generating and quantifying various types of entanglement in atom-photon interactions governed by the Jaynes-Cummings model.

References

[1] A. Einstein, B. Podolsky and N. Rosen, Phys. Rev. **47**, 777 (1935).
[2] J. S. Bell, Physics **1**, 195 (1964). A text book presentation is given by L. E. Ballentine, *Quantum Mechanics*, (World Scientific, 2002).
[3] E. Schrödinger, Proc. Camb. Phil. Soc. **31**, 555 (1935).
[4] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Information* (Cambridge University Press, Cambridge, England, 2000).
[5] C. H. Bennett, and P. W. Shor, IEEE Trans. Inform. Theory **44**, 2724 (1998).
[6] S. Roy and B. Ghosh, *Int. Jour. of Quant. Inf*, **13**, No. 5, 1550033, (2015).
[7] J. M. Raimond, M. Brune, and S. Haroche, Rev. Mod. Phys. **73**, 565 (2001).
[8] T. Tessier, A. Delgado, I. Fuentes-Guridi, and I. H. Deutsch, Phys. Rev. A **68**, 062316 (2003).
[9] P. Masiak, Phys. Rev. A **66**, 023804 (2002).
[10] A. Datta, B. Ghosh, A. S. Majumdar and N. Nayak, Europhys. Lett. **67**, 934 (2004).
[11] M. S. Kim, Jinhyoung Lee, D. Ahn and P. L. Knight, Phys. Rev. A **65**, 040101(R) (2002).
[12] L. Zhou, H. S. Song and C. Li, J. Opt. B: Quantum Semiclass. Opt. **4**, 425 (2002).
[13] B. Ghosh, A. S. Majumdar and N. Nayak, Int. J. Quant. Inf. **5**, 169 (2007).
[14] B. Ghosh, A. S. Majumdar and N. Nayak, Int. J. Theo. Phys., Group Theo., and Nonl. Opt. **12**, Issue 2, 1-11 (2007).
[15] E. T. Jaynes, F. W. Cummings, Proc. IEEE **51**, 89 (1963).
[16] L. Allen and J. H. Ebarly, *Optical Resonance and Two-level Atoms* (Dover Publications, 1987).
[17] S. J. D. Phoenix and P. L. Knight Phys. Rev. A **44**, 6023 (1991).
[18] S. Furuichi, and M. Ohya, Lett. Math. Phys. **49**, 279 (1999).
[19] S. Bose, I. Fuentes-Gurudi, P. L. Knight, and V. Vedral, Phys. Rev. Lett. **87**, 050401 (2001).
[20] See, for instance, W. H. Louisell, *Quantum Statistical Properties of Radiation* (Wiley, New York, 1973).
[21] G. S. Agarwal, in *Springer Tracts in Modern Physics*, Vol.70, (Springer-Verlag, Berlin & New York, 1974).
[22] S. Hill and W. K. Wootters, Phys. Rev. Lett. **78**, 5022, (1997).
[23] W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).
[24] W. K. Wootters, Quantum Inf. Comput. **1**, 2744, (2001).
[25] G. Vidal and R. F. Werner, Phys. Rev. A **65**, 32314, (2002).
[26] M. B. Plenio, Phys. Rev. Lett. **95**, 090503, (2005).
[27] S. Haroche and J. M. Raimond, in "*Advances in atomic and molecular physics*", Vol. 20, eds. D. R. Bates and B. Bederson (Academic, New York, 1985).
[28] T. Yu, J.H. Eberly, Phys. Rev. Lett. **85**, 140404 (2004).

Biplab Ghosh, received his B.Sc. Degree from Calcutta University, Kolkata and M.Sc. Degree from University of Kalyani. After completing his post graduation, he was engaged in research work in the field of Quantum Information Theory at S. N. Bose National Centre for Basic Sciences, Kolkata, India. He got his Ph. D. Degree from Jadavpur University and presently engaged in teaching in UG physics.