Mathematical Sciences Letters
An International Journal

http://dx.doi.org/10.18576/msl/060107

Some Well-Known Fixed Point Theorems in Dislocated Quasi-b-Metric Space

*Mujeeb Ur Rahman**

Department of Mathematics, Government PG Jahanzeb College Saidu Sharief Swat, Khyber PakhtunKhwa, Pakistan.

Received: 18 Jan. 2016, Revised: 17 Sep. 2016, Accepted: 20 Sep. 2016
Published online: 1 Jan. 2017

Abstract: In this article, some well-known fixed point theorems like φ-contraction and Reich type contraction are established in the frame work of dislocated quasi-b-metric space. Several corollaries are deduced from the theorems which extend and generalize some well-known results in the literature. Examples are given in order to validate our main results.

Keywords: Complete dislocated quasi-b-metric space, Cauchy sequence, self-mapping, fixed point.

1 Introduction

Fixed point theory is one of the most important topic in the development of non-linear analysis also fixed point theory is widely applicable in many branches of science such as Chemistry, Biology, Economics, Computer Science and Engineering etc.

One branch of generalizations of celebrated Banach contraction principle is based on the replacement of contraction condition imposed on $T : X \to X$, where (X,d) is a complete metric space. The weaker condition described by Browder [1] as, $d(Tx,Ty) \le \varphi d(x,y)$ for all $x,y \in X$, where φ is a comparison function introduced by Berinde [2]. Reich [3] generalized the Banach contraction principle by introducing a new type of contraction condition which were given the name of Reich type contraction. In similar direction Istratescu [4] introduced the convex type contraction and generalized Banach contraction principle for such a type of contraction condition.

The notion of b-metric space was introduced by Czerwik [5] in connection with some problems concerning with the convergence of non-measurable functions with respect to measure. Fixed point theorems regarding b-metric spaces was obtained in [6] and [7]. In 2013, Shukla [8] generalized the notion of b-metric spaces and introduced the concept of partial b-metric spaces. Recently, Rahman and Sarwar [9] further generalized the concept of b-metric space and initiated the notion of dislocated quasi-b-metric space.

Motivated by the above work, we have proved φ-contraction and Reich type of contraction in the setting of dislocated quasi-b-metric space which generalize and extend Banach contraction principle and convex type contraction in dislocated quasi-b-metric space.

2 Preliminaries

We need the following definitions which may be found in [9].

Definition 1. Let X be a non-empty set and $k \ge 1$ be a real number then a mapping $d : X \times X \to [0,\infty)$ is called dislocated quasi-b-metric if $\forall\, x,y,z \in X$

(d_1) $d(x,y) = d(y,x) = 0$ implies that $x = y$;

(d_2) $d(x,y) \le k[d(x,z) + d(z,y)]$.

The pair (X,d) is called dislocated quasi-b-metric space or shortly (dq b-metric) space.

Remark. In the definition of dislocated quasi-b-metric space if $k = 1$ then it becomes (usual) dislocated quasi metric space. Therefore every dislocated quasi metric space is dislocated quasi-b-metric space and every b-metric space is dislocated quasi-b-metric space with same coefficient k and zero self distance. However, the converse is not true as clear from the following example.

Example 1. Let $X = \mathbb{R}$ and suppose

$$d(x,y) = |2x-y|^2 + |2x+y|^2.$$

Then (X,d) is a dislocated quasi-b-metric space with the coefficient $k = 2$. But it is not dislocated quasi-metric

* Corresponding author e-mail: mujeeb846@yahoo.com

space nor b-metric space.
Remark. Like dislocated quasi metric space in dislocated quasi-b-metric space the distance between similar points need not to be zero necessarily as clear from the above example.
Definition 2. A sequence $\{x_n\}$ is called dq-$b-$convergent in (X,d) if for $n \geq N$ we have $d(x_n,x) < \varepsilon$ where $\varepsilon > 0$ then x is called the dq-b-limit of the sequence $\{x_n\}$.
Definition 3. A sequence $\{x_n\}$ in dq-b-metric space (X,d) is called Cauchy sequence if for $\varepsilon > 0$ there exists $n_0 \in N$, such that for $m,n \geq n_0$ we have $d(x_m,x_n) < \varepsilon$.
Definition 4. A dq-$b-$metric space (X,d) is said to be complete if every Cauchy sequence in X converges to a point of X.
The following well-known results can be seen in [9].
Lemma 1. Limit of a convergent sequence in dislocated quasi-b-metric space is unique.
Lemma 2. Let (X,d) be a dislocated quasi-b-metric space and $\{x_n\}$ be a sequence in dqb-metric space such that

$$d(x_n,x_{n+1}) \leq \alpha d(x_{n-1},x_n) \tag{1}$$

for $n = 1,2,3,...$ and $0 \leq \alpha k < 1, \alpha \in [0,1)$, and k is defined in dq-b-metric space. Then $\{x_n\}$ is a Cauchy sequence in X.
Theorem 1. Let (X,d) be a complete dislocated quasi-b-metric space. Let $T : X \to X$ be a continuous contraction with $\alpha \in [0,1)$ and $0 \leq k\alpha < 1$ where $k \geq 1$. Then T has a unique fixed point in X.
Remark. Like b-metric space dislocated quasi-b-metric space is also continuous on its two variables.
Theorem 2.[10]. Every φ-contraction $T : X \to X$ where (X,d) is a complete metric space, is a Picard's operator.
Definition 5.[2]. A map $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is called comparison function if it satisfies:

1. φ is monotonic increasing;
2. The sequence $\{\varphi^n(t)\}_{n=0}^{\infty}$ converge to zero for all $t \in \mathbb{R}_+$ where φ^n stand for nth iterate of φ.
   If φ satisfies:
3. $\sum_{k=0}^{\infty} \varphi^k(t)$ converge for all $t \in \mathbb{R}_+$.

Then φ is called (c)-comparison function.
Thus every comparison function is c-comparison function. A prototype example for comparison function is

$$\varphi(t) = \alpha t \quad t \in \mathbb{R}_+ \quad 0 \leq \alpha < 1.$$

Some more examples and properties of comparison and c-comparison function can be found in [2].

3 Main Results

Theorem 3.1. Let (X,d) be a complete dislocated quasi-b-metric space. Let $T : X \to X$ be a continuous function for $k \geq 1$ satisfying

$$d(Tx,Ty) \leq \varphi d(x,y) \tag{2}$$

for all $x,y \in X$ where φ is a comparison function. Then T has a unique fixed point in X.
Proof. Let x_0 be arbitrary in X we define a sequence $\{x_n\}$ in X as following

$$x_0, x_1 = Tx_0, x_2 = Tx_1, ..., x_{n+1} = Tx_n \text{ for all } n \in \mathrm{N}.$$

Now to show that $\{x_n\}$ is a Cauchy sequence in X consider

$$d(x_n,x_{n+1}) = d(Tx_{n-1},Tx_n).$$

Using (2) we have

$$d(x_n,x_{n+1}) \leq \varphi d(x_{n-1},x_n). \tag{3}$$

Similarly one can show that

$$d(x_{n-1},x_n) \leq \varphi d(x_{n-2},x_{n-1}). \tag{4}$$

Putting (3) in (4) we have

$$d(x_n,x_{n+1}) \leq \varphi^2 d(x_{n-2},x_{n-1}).$$

Proceeding in similar manner we get

$$d(x_{n-1},x_n) \leq \varphi^n d(x_0,x_1). \tag{5}$$

To show that $\{x_n\}$ is a Cauchy sequence consider $m > n$ and using (d_2) we have

$$d(x_n,x_m) \leq k \cdot d(x_n,x_{n+1}) + k^2 \cdot d(x_{n+1},x_{n+2}) + k^3 \cdot d(x_{n+2},x_{n+3}) + ...$$

Using (5) the above equation become

$$d(x_n,x_m) \leq k \cdot \varphi^n d(x_0,x_1) + k^2 \cdot \varphi^{n+1} d(x_0,x_1) + k^3 \cdot \varphi^{n+2} d(x_0,x_1) + ...$$

Since φ is a comparison function so taking $n,m \to \infty$ we get

$$\lim_{n,m\to\infty} d(x_n,x_m) = 0.$$

Which show that $\{x_n\}$ is a Cauchy sequence in complete dislocated quasi-b-metric space X. So there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$.
Now to show that z is the fixed point of T. Since $x_n \to z$ as $n \to \infty$ using the continuity of T we have

$$\lim_{n\to\infty} Tx_n = Tz$$

which implies that

$$\lim_{n\to\infty} x_{n+1} = Tz.$$

Thus $Tz = z$. So z is the fixed point of T.
Uniqueness: Suppose that T has two fixed points z and w for $z \neq w$. Consider

$$d(z,w) = d(Tz,Tw).$$

Using (2) we have

$$d(z,w) \le \varphi d(z,w).$$

Since φ is a comparison function so the above inequality is possible only if $d(z,w) = 0$ similarly one can show that $d(w,z) = 0$. So by (d_1) $z = w$. Hence T has a unique fixed point in X.

Remark. Theorem 3.1 generalize the Banach contraction principle and the result established by Matkowski [10] in dislocated quasi-b-metric spaces.

Example 3.1. Let $X = [0,1]$ and $T : X \to X$ is defined as

$$Tx = \frac{x}{6} \ \ \forall\, x \in X$$

with complete dislocated quasi-b-metric space is given by

$$d(x,y) = |2x-y|^2 + |2x+y|^2 \ \ \forall\, x,y \in X.$$

Then

$$d(Tx,Ty) = d(\frac{x}{6},\frac{y}{6}) = |\frac{x}{3} - \frac{y}{6}|^2 + |\frac{x}{3} + \frac{y}{6}|^2$$

$$d(Tx,Ty) = d(\frac{x}{6},\frac{y}{6}) = \frac{1}{36}(|2x-y|^2 + |2x+y|^2)$$
$$\le \frac{1}{4}(|2x-y|^2 + |2x+y|^2).$$

Hence

$$d(Tx,Ty) \le \varphi d(x,y) \ \text{ where } \varphi(t) = \frac{t}{4} \ \text{ where } t \in X.$$

Thus T has a unique fixed point $0 \in [0,1]$.

Theorem 3.2. Let (X,d) be a complete dq-b-metric space and $T : X \to X$ is a continuous self-mapping satisfying

$$d(Tx,Ty) \le \alpha \cdot d(x,y) + \beta \cdot d(x,Tx) + \gamma \cdot d(y,Ty) \tag{6}$$

for all $x,y \in X$ and $\alpha,\beta,\gamma \ge 0$ with $k\alpha + k\beta + \gamma < 1$ where $k \ge 1$. Then T has a unique fixed point in X.

Proof. Let x_0 be arbitrary in X we define a sequence $\{x_n\}$ in X as following

$$x_0, x_1 = Tx_0, x_2 = Tx_1, \ldots\ldots\ldots\ldots, x_{n+1} = Tx_n.$$

Now to show that $\{x_n\}$ is a Cauchy sequence consider

$$d(x_n,x_{n+1}) = d(Tx_{n-1},Tx_n).$$

Using (6) we have

$$d(x_n,x_{n+1}) = d(Tx_{n-1},Tx_n) \le \alpha \cdot d(x_{n-1}+x_n) +$$
$$\beta \cdot d(x_{n-1},Tx_{n-1}) + \gamma \cdot d(x_n,Tx_n).$$

By the definition of the sequence we get

$$d(x_n,x_{n+1}) \le \alpha \cdot d(x_{n-1},x_n) + \beta \cdot d(x_{n-1},x_n) +$$
$$\gamma \cdot d(x_n,x_{n+1}).$$

Simplification yields

$$d(x_n,x_{n+1}) \le \frac{\alpha+\beta}{1-\gamma} \cdot d(x_{n-1},x_n).$$

Let

$$h = \frac{\alpha+\beta}{1-\gamma} < \frac{1}{k}.$$

So the above inequality become

$$d(x_n,x_{n+1}) \le h \cdot d(x_{n-1},x_n).$$

Also

$$d(x_{n-1},x_n) \le h \cdot d(x_{n-2},x_{n-1}).$$

Thus

$$d(x_n,x_{n+1}) \le h^2 \cdot d(x_{n-2},x_{n-1}).$$

Similarly proceeding we get

$$d(x_n,x_{n+1}) \le h^n \cdot d(x_0,x_1).$$

Since $h < \frac{1}{k}$. Taking limit $n \to \infty$, so $h^n \to 0$ and

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0.$$

So by Lemma 2 $\{x_n\}$ is a Cauchy sequence in complete dq-b-metric space so there must exist $u \in X$ such that

$$\lim_{n\to\infty} (x_n,u) = 0.$$

Now to show that u is the fixed point of T. Since $x_n \to u$ as $n \to \infty$ using the continuity of T we have

$$\lim_{n\to\infty} Tx_n = Tu$$

which implies that

$$\lim_{n\to\infty} x_{n+1} = Tu.$$

Thus $Tu = u$. So u is the fixed point of T.

Uniqueness: Let T have two fixed points i.e u,v with $u \ne v$ then we have

$$d(u,v) = d(Tu,Tv) \le \alpha \cdot d(u,v) + \beta \cdot d(u,Tu) + \gamma \cdot d(v,Tv)$$

$$d(u,v) = d(Tu,Tv) \le \alpha \cdot d(u,v) + \beta \cdot d(u,u) + \gamma \cdot d(v,v).$$

Putting $u = v$ in (6) one can easily show that $d(u,u) = d(v,v) = 0$. Thus the above equation become

$$d(u,v) \le \alpha \cdot d(u,v).$$

The above inequality is possible only if $d(u,v) = 0$ similarly one can show that $d(v,u) = 0$. So by (d_1) we get that $u = v$. Thus fixed point of T is unique.

Corollary 3.1. Let (X,d) be a complete dq-b-metric space and $T : X \to X$ is a continuous self-mapping satisfying

$$d(Tx,Ty) \le \alpha \cdot d(x,y) + \beta \cdot d(x,Tx)$$

for all $x,y \in X$ and $\alpha,\beta \geq 0$ with $k\alpha + k\beta < 1$ where $k \geq 1$. Then T has a unique fixed point in X.

Corollary 3.2. Let (X,d) be a complete dq-b-metric space and $T : X \to X$ is a continuous self-mapping satisfying

$$d(Tx,Ty) \leq \alpha \cdot d(x,y)$$

for all $x,y \in X$ and $\alpha \geq 0$ with $0 \leq k\alpha < 1$ where $k \geq 1$. Then T has a unique fixed point in X.

Remark. Theorem 3.2 generalize Reich type contraction and extend Banach contraction principle and convex type contraction in complete dislocated quasi-b-metric spaces.

Example 3.2. Let $X = [0,1]$ and $T : X \to X$ is defined as

$$Tx = \frac{x}{6} \ \ \forall\, x \in X$$

with complete dislocated quasi-b-metric space is given by

$$d(x,y) = |2x-y|^2 + |2x+y|^2 \ \ \forall\, x,y \in X.$$

Then

$$d(Tx,Ty) = d(\frac{x}{6},\frac{y}{6}) = |\frac{x}{3}-\frac{y}{6}|^2 + |\frac{x}{3}+\frac{y}{6}|^2$$

$$d(Tx,Ty) = d(\frac{x}{6},\frac{y}{6}) = \frac{1}{36}(|2x-y|^2+|2x+y|^2) \leq \frac{1}{6}(|2x-y|^2+|2x+y|^2).$$

Hence

$$d(Tx,Ty) \leq \alpha \cdot d(x,y).$$

Thus T has a unique fixed point $0 \in [0,1]$.

4 Conclusion

The results established in this paper generalize, extend and improved the results of Banach contraction principle, Matkowski [10], Reich type contraction and convex type contraction in complete dislocated quasi b-metric space.

Acknowledgement

The author is thankful to the editor in chief and anonymous referees for there valuable remarks and suggestions which enable us to improve this article.

References

[1] P.E. Browder, On the convergence of succesive approximations for non-linear functional equations, *Nedelr. Akad. Wet. Proc. Ser.*, **30**(1968), 27-35.

[2] V. Berinde, On the approximation of fixed points of weak contractive mappings, *Carpathian Journal of Mathematics*, **19**(2003), 722.

[3] S. Reich, Fixed points of contractive functions, *Bullettino DellUnione Mathematica Italiana*, **5**(1972), 26 42.

[4] V. Istratescu, Some fixed point theorems for convex contraction mappings and convex nonexpansive mappings (I), *Libertas Math.*, **1**(1981), 151-164.

[5] S. Czerwik, Contraction mappings in *b*-metric spaces, *ActaMath.Inform. Univ. Ostraviensis*, **1**(1993), 5-11.

[6] M. Sarwar and M.U. Rahman, Fixed point theorems for Ciric's and generalized contractions in *b*-metric spaces,*International Journal of Analysis and Applications*, **7**(2015), 70-78.

[7] M. Kir and H. Kizitune, On some well-known fixed point theorems in *b*-Metric Space, *Turkish Journal of Analysis and Number Theory*, **1**(2013), 13-16.

[8] S. Shukla, Partial *b*-metric spaces and fixed point theorems, *Mediterr. J. Math.*, June(2013).

[9] M.U. Rahman and M. Sarwar, Dislocated quasi *b*-metric space and fixed pont theorems, *Electronic Journal of Mathematical Analysis and Applications*, **4**(2016).

[10] J. Matkowski, Integrable soloution of functional equations, *Dissertation Math. (Rozprawy)*, **127**(1976).

[11] P.S. Kumari, D. Panthi, Cyclic contraction and fixed point theorems on various generating spaces, *Fixed Point Theory and Applications*, **153**(2015), 2015.

[12] P.S. Kumari, On dislocated quasi-metrics, *Journal of Advance Studies in Topology*, **3**(2012), 66-75.

[13] M.U. Rahman, M. Sarwar, P.S. Kumari, Common tripled fixed point theorem indislocated quasi-metric spaces, *Sindh University Research Journal*, **47**(2015), 1-4.

Mujeeb Ur Rahman Completed has M-Phil degree from University of Malakand, Dir(L), Khyber Pakhtunkhwa, Pakistan in 2014. Presently working as a lecturer of Mathematics in Government PG Jahanzeb College Swat, Pakistan. His areas of interest are Functional Analysis and Metric Fixed Point Theory. He has published eighteen research papers in various reputed international journals of Mathematics. He is referee of several well-known journals of Mathematics.