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# An Efficient Modification of the Adomian decomposition Method for Solving Integro-Differential Equations

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**Abstract:** In this paper, an application of new modification of Adomian decomposition method is applied to solve integro-differential equations. The exact solutions are represented in simple form. Moreover, the convergence to the exact solutions is accelerated. Some examples are given to illustrate the ability and reliability of this new modified method.

**Keywords:** Volterra integro-differential equation, Fredholm integro-differential equation, Adomian decomposition method, New modification, System of integro-differential equation.

## 1 Introduction

Integral equations and differential equations play an important role in many social, biological, physical and engineering problems, as can be seen from the attached references ([1]-[15]). Analytical methods for the solution of nonlinear Integro-differential equations are usually hard ([16]-[17]), if not impossible, consequently exact solutions are rather difficult to be obtained. Therefore, several numerical methods are used for the solution of such types of equations such as the finite differences [18], Tau method [19], He's homotopy perturbation method ([20]-[25]), Legendre polynomials and Block-pulse functions approach [26], Haar Wavelet Series Technique [27], differential transform ([28]-[29]) and the Adomian decomposition method ([30]-[31]). For the solution of integral equations, Adomian presented the so-called Adomian decomposition method (ADM) ([32]-[33]). Wazwaz extended the method to include the solution of Volterra integral equation [34] and the boundary value problems for higher order integro-differential equations. In recent years, much work has been concentrated on the solutions of Volterra-Fredholm integro-differential equation, in general, ([35]-[39]).

Adomian's decomposition method (ADM), as a powerful method, can be used to solve all types of nonlinearity. At the same time, this method reduces the size of computation, while increasing the accuracy of the solutions. The ADM separates the equation to be solved into two portions: linear and nonlinear. The solution generated by this method is in a series form whose terms are determined by a recursive relation using Adomian polynomials

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}N\left(\sum_{i=0}^{\infty}\lambda_i u_i\right)\right]_{\lambda=0}, \qquad n \geq 0. \tag{1}$$

Where $A_n$ denotes the Adomian polynomials of degree $n$ and $u = \sum_{i=0}^{\infty} u_i(x,t)$ is the solution of the problem and $N(u)$ is the nonlinear term in the equation.

Recently, a new modification of Adomian decomposition method (NMADM) for finding exact solutions of nonlinear integral equations is presented by Hossein Jafari, et al. [40]. A new reliable modification of the ADM is proposed and applied for the solution of the Volterra and Fredholm integral equations. In this paper, the (NMADM) is applied for the solution of nonlinear integro-differential equations (NIDE). Some examples are given to illustrate the ability and reliability of this new modified method. The results reveal that the presented modified method is very simple and effective.

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## 2 Adomian decomposition method for solving (NIDE)

The nonlinear Volterra integro-differential equations of the second kind are given by

$$u''(x) = f(x) + \int_a^x K(x,t)\,[R(u(t)) + N(u(t))]\,dt,$$
$$u(x_0) = a_1, \qquad u'(x_0) = a_2, \tag{2}$$

and the nonlinear Fredholm integro-differential equations the second kind are given by

$$u''(x) = f(x) + \int_a^b K(x,t)\,[R(u(t)) + N(u(t))]\,dt,$$
$$u(x_0) = a_1, \qquad u'(x_0) = a_2. \tag{3}$$

Where $u^{''}(x)$ is the second derivative of the unknown function $u(x)$ that will be determined, $K(x,t)$ is the kernel of the integral equation, $f(x)$ is an analytic function, $R(u(t))$ and $N(u(t))$ are linear and nonlinear function of $u$, respectively. Equation (1) and (2) can be rewritten in an operator form

$$L(u(x)) = f(x) + \int_a^x K(x,t)\,[R(u(t)) + N(u(t))]\,dt, \tag{4}$$

$$L(u(x)) = f(x) + \int_a^b K(x,t)\,[R(u(t)) + N(u(t))]\,dt. \tag{5}$$

The inverse operator $L^{-1}$ is therefore considered an $n-$fold integral operator defined by $L_x^{-1}(.) = \int_0^x (.)dx$. Operating with $L_x^{-1}$ to both sides of (4) and (5) and using the initial condition, we have

$$u(x) = g(x) + L_x^{-1}\Big(\int_a^x K(x,t)\,[R(u(t)) + N(u(t))]\,dt\Big), \tag{6}$$

and

$$u(x) = g(x) + L_x^{-1}\Big(\int_a^b K(x,t)\,[R(u(t)) + N(u(t))]\,dt\Big), \tag{7}$$

where $g(x)$ included the $L_x^{-1}(f(x))$ and the initial conditions.

The standard Adomian decomposition method defines the solution $u(x)$ by the decomposition series

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \tag{8}$$

where $u_n(x)$ has to be determined sequentially upon the following algorithm

$$u_0(x) = g(x) \tag{9}$$

$$u_{m+1}(x) = L_x^{-1}\Big(\int_a^b K(x,t)\,[R(u_m(t)) + A_m(u(t))]\,dt\Big),$$
$$m \geq 0, \tag{10}$$

where $A_m$, $m \geq 0$ are the Adomian polynomials.

## 3 A new Adomian decomposition method for solving (NIDE)

The modified Adomian decomposition method (NMADM) provides the exact solution by using a single iteration. Only $u_0$ and $u_1$, which give one exact solution(s) after one iteration, are discussed. The solution is usually a unique solution, but in the present work, an example that gives two solutions is presented. In this method, the rate of convergence is accelerated. The ADM usually gives one solution among other solutions. However, as will be seen in an example below, the presented method can give more than one solution". To achieve this goal, Equations (6) and (7) are rewritten as:

$$u(x) = \sum_{m=0}^{N} a_m v_m(x) - \sum_{m=0}^{N} a_m v_m(x) + g(x) + \\ + L_x^{-1}\Big(\int_a^x K(x,t)\,[R(u(t)) + N(u(t))]\,dt\Big), \tag{11}$$

and

$$u(x) = \sum_{m=0}^{N} a_m v_m(x) - \sum_{m=0}^{N} a_m v_m(x) + g(x) + \\ + L_x^{-1}\Big(\int_a^b K(x,t)\,[R(u(t)) + N(u(t))]\,dt\Big), \tag{12}$$

where $a_m$, $m = 0,1,2,...,N$ are called the accelerating components of the parameter, and $v_m(x)$, $m = 0,1,2,...,N$ are selective functions. Furthermore, the number of the terms in $u_0$, namely $N$, is small in many practical problems. Recall that the modified decomposition method is established based on the assumption that the function

$$f(x) = \sum_{m=0}^{N} a_m v_m(x) - \sum_{m=0}^{N} a_m v_m(x) + g(x), \tag{13}$$

can be divided into two parts, namely $f_1(x)$ and $f_2(x)$. Under this assumption we set

$$f(x) = f_1(x) + f_2(x),$$

where

$$f_1(x) = \sum_{m=0}^{N} a_m v_m(x),$$

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and

$$f_2(x) = -\sum_{m=0}^{N} a_m v_m(x) + g(x).$$

Accordingly, a slight variation was proposed only on the components $u_0$ and $u_1$. It is suggested that only the part $f_1(x)$ be assigned to the zeroth component $u_0$, whereas the remaining part $f_2(x)$ be combined with the other terms given in (11) and (12) to define $u_1$. Consequently, the modified recursive relation:

$$u_0(x) = \sum_{m=0}^{N} a_m v_m(x), \tag{14}$$

$$u_1(x) = -\sum_{m=0}^{N} a_m v_m(x) + g(x) + L_x^{-1}\left(\int_a^b K(x,t)\left[R(u_0(t)) + A_0(u(t))\right]dt\right), \tag{15}$$

$$u_m(x) = L_x^{-1}\left(\int_a^b K(x,t)\left[R(u_{m-1}(t)) + A_{m-1}(u(t))\right]dt\right), \quad m \geq 2. \tag{16}$$

was developed.

## 4 Numerical illustrations

In this section, several examples are solved to illustrate this method for Linear and Nonlinear Integro-differential equations.

### 4.1 Linear integro-differential equations

*Example 1.*Consider the linear Fredholm integro-differential equation

$$u'(x) = xe^x + e^x - x + \int_0^1 xu(t)dt, \quad u(0) = 0, \tag{17}$$

with the exact solution $u(x) = xe^x$. $u_0(x)$ is chosen as:

$$u_0(x) = \sum_{m=0}^{1} a_m x^m e^x = a_0 e^x + a_1 x e^x.$$

In view of Eq. (15) we have

$$u_1(x) = -a_0 e^x - a_1 x e^x - 1/2x^2 + xe^x + \int_0^x \left(\int_0^1 xu_0(t)dt\right)dx = 0$$

Now, $a_m$, $m = 0,1$ are found such that $u_1(x) = 0$. If $u_1(x) = 0$ then $u_2(x) = u_3(x) = u_4(x) = \cdots = 0$, and the exact solution will be obtained as $u(x) = u_0(x)$. Hence for all values of $x$ we have

$$e^x x - \frac{1}{2}x^2 - e^x a_0 - \frac{1}{2}x^2 a_0 + \frac{1}{2}ex^2 a_0 - e^x x a_1 + \frac{1}{2}x^2 a_1 = 0$$

$-a_0 = 0$
$a_1 = 1$
So, the solutions will be $u(x) = xe^x$, which is the same as the exact solution.

*Example 2.*Consider the linear Volterra integro-differential equation

$$u''(x) = 1 + x + \int_0^x (x-t)u(t)dt, u(0) = 1, u'(0) = 1. \tag{18}$$

$u_0(x)$ are chosen as:

$$u_0(x) = \sum_{m=0}^{1} a_m x^m = a_0 + a_1 x$$

In view of Eq. (15) we have

$$u_1(x) = -a_0 - a_1 x + 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \int_0^x \int_0^x \left(\int_0^x (x-t)u_0(t)dt\right)dxdx = 0,$$

which means that:

$$-a_0 - a_1 x + \frac{x^4 a_0}{24} + \frac{x^5 a_1}{120} + 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} = 0$$

Now, $a_m$, $m = 0,1$ are found such that $u_1(x) = 0$. Hence, $a_0 = 1$, $a_1 = 1$, for all values of $x$. So, the solutions will be $u(x) = 1 + x$, which satisfies the conditions, but does not satisfy the equation, by substituting this solution in the right hand side of equation (18) we get;
R.H.S.$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$ So, we increase $N$ and let $u_0(x)$ to be

$$u_0(x) = \sum_{m=0}^{2} a_m x^m = a_0 + a_1 x + a_2 x^2$$

In view of Eq. (15) we have

$$u_1(x) = -a_0 - a_1 x - a_2 x^2 + 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \int_0^x \int_0^x \left(\int_0^x (x-t)u_0(t)dt\right)dxdx = 0$$

which means that

$$-a_0 - a_1 x - a_2 x^2 + 1 + x + x^2 + \frac{x^3}{2} + \frac{x^4 a_0}{24} + \frac{x^5 a_1}{120} + \frac{x^6 a_2}{360} = 0$$

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Now, $a_m$, $m = 0,1,2$ are found such that $u_1(x) = 0$. Hence for all values of $x$ we have $a_0 = 1$, $a_1 = 1$, $a_2 = \frac{1}{2}$. So, the solutions will be $u(x) = 1 + x + \frac{x^2}{2}$, which also satisfies the conditions, but does not satisfy the equation, by substituting this solution in the right hand side of equation (18) we get
R.H.S.$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{12}$. So, we increase $N$ and let

$$u_0(x) = \sum_{m=0}^{\infty} a_m x^m,$$

$$u_1(x) = -\sum_{m=0}^{\infty} a_m x^m + 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} +$$
$$+ \int_0^x \int_0^x \Big( \int_0^x \Big( (x-t) \sum_{m=0}^{\infty} a_m t^m \Big) dt \Big) dx dx.$$

Therefore, the exact solution is suggested to be $u(x) = e^x$, which satisfies both the conditions, and the equation.

## 4.2 Nonlinear integro-differential equations

*Example 3.*Consider the nonlinear Fredholm integro-differential equation

$$u' = \frac{5}{4} - \frac{1}{3}x^2 + \int_0^1 (x^2 - t)\big(u(t)\big)^2 dt, u(0) = 0, \qquad (19)$$

with the exact solution $u(x) = x$. $u_0(x)$ is chosen to be:

$$u_0(x) = \sum_{m=0}^{1} a_m x^m = a_0 + a_1 x$$

In view of Eq. (15) we have

$$u_1(x) = -a_0 - a_1 x + \frac{5x}{4} - \frac{x^3}{9} +$$
$$+ \int_0^x \Big( \int_0^1 (s^2 - t) A_0(t) dt \Big) ds = 0$$

where $A_m$ are the well-known Adomian polynomials. The first term of Adomian polynomials that represents the nonlinear operator $u^2(x)$ is $A_0(x) = u_0^2(x)$. Now we find $a_m$, $m = 0,1$. Hence for all values of $x$ we have

$$\frac{5x}{4} - \frac{x^3}{9} - a_0 - \frac{xa_0^2}{2} + \frac{1}{3}x^3 a_0^2 - xa_1 - \frac{2}{3}xa_0 a_1 +$$
$$+ \frac{1}{3}x^3 a_0 a_1 - \frac{xa_1^2}{4} + \frac{1}{9}x^3 a_1^2 = 0,$$

which means that
$a_0 = 0$,
$\frac{5}{4} - \frac{a_0^2}{2} - a_1 - \frac{2}{3}a_0 a_1 - \frac{a_1^2}{4} = 0$ or $a_1 = 1, -5$.
So, the solution, which is the same as the exact solution and satisfies both the equation and the conditions, will be $u(x) = x$, while, the solution $u(x) = -5x$ does not.

*Example 4.*Consider the nonlinear Volterra integro-differential equation

$$u' = -1 + \int_0^x u(t)^2 dt, \quad u(0) = 0, \qquad (20)$$

with the exact solution $u(x) = -x$. We can choose $u_0(x)$ to be

$$u_0(x) = \sum_{m=0}^{1} a_m x^m = a_0 + a_1 x.$$

In view of Eq. (15) we have

$$u_1(x) = -a_0 - a_1 x - x + \int_0^x \Big( \int_0^x A_0(t) dt \Big) dx = 0,$$

where $A_0$ is the Adomian polynomial. Now, $a_m$, $m = 0,1$ are found from the relation:

$$-a_0 - a_1 x - x + \frac{1}{2}x^2 a_0^2 + \frac{1}{3}x^3 a_0 a_1 + \frac{1}{12}x^4 a_1^2 = 0,$$

which gives:
$a_0 = 0$,
$a_1 = -1$.
So, the solution will be $u(x) = -x$.

## 4.3 Nonlinear system of integro-differential equations

*Example 5.*Consider the nonlinear system of Fredholm integro-differential equations

$$u_1' = e^x - 12 + \int_0^{Ln2} \Big( u_1(t)^2 + u_2(t)^2 \Big) dt, u_1(0) = 1,$$
$$u_2' = 3e^{3x} + 9 + \int_0^{Ln2} \Big( u_1(t)^2 - u_2(t)^2 \Big) dt, u_2(0) = 1, \qquad (21)$$

with the exact solution $u_1(x) = e^x$, $u_2(x) = e^{3x}$. $u_0(x)$ is chosen as:

$$u_{10}(x) = \sum_{m=0}^{2} a_m e^{(m+1)x} = a_0 e^x + a_1 e^{2x} + a_2 e^{3x},$$

$$u_{20}(x) = \sum_{m=0}^{2} b_m e^{(m+1)x} = b_0 e^x + b_1 e^{2x} + b_2 e^{3x}.$$

In view of Eq. (15) we have

$$u_{11}(x) = -a_0 e^x - a_1 e^{2x} - a_2 e^{3x} + e^x - 12x +$$
$$+ \int_0^x \Big( \int_0^{Ln2} (A_0[t] + B_0[t] \Big) dt) ds = 0$$
$$u_{21}(x) = -b_0 e^x - b_1 e^{2x} - b_2 e^{3x} + e^{3x} + 9x +$$
$$+ \int_0^x \Big( \int_0^{Ln2} (A_0[t] - B_0[t] \Big) dt) ds = 0$$

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where $A_0$ and $B_0$ are the Adomian polynomials. Now, $a_m$, $m = 0,1,2$ are found from the relation:

$$e^x - 12x + (\frac{3}{2}a_0^2 + \frac{14}{3}a_0a_1 + \frac{15}{2}a_0a_2 +$$
$$+\frac{15}{4}a_1^2 + \frac{62}{6}a_1a_2 + \frac{21}{2}a_2^2 + \frac{3}{2}b_0^2 +$$
$$+\frac{14}{3}b_0b_1 + \frac{15}{2}b_0b_2 + \frac{15}{4}b_1^2 + \frac{62}{5}b_1b_2 +$$
$$+\frac{21}{2}b_2^2)x - a_0e^x - a_1e^{2x} - a_2e^{3x} = 0$$

as:
$a_0 = 1$,
$a_1 = a_2 = 0$.
Similarly, $b_m$, $m = 0,1,2$ are found from the relation:

$$e^{3x} + 9x + (\frac{x^3}{2}a_0^2 + \frac{14}{3}a_0a_1 + \frac{15}{2}a_0a_2 +$$
$$+\frac{15}{4}a_1^2 + \frac{62}{6}a_1a_2 + \frac{21}{2}a_2^2 - \frac{3}{2}b_0^2 -$$
$$-\frac{14}{3}b_0b_1 - \frac{15}{2}b_0b_2 - \frac{15}{4}b_1^2 - \frac{62}{5}b_1b_2 -$$
$$-\frac{21}{2}b_2^2)x - b_0e^x - b_1e^{2x} - b_2e^{3x} = 0$$

as
$b_0 = b_1 = 0$,
$b_2 = 1$.
So, the solution will be $u_1(x) = e^x$ and $u_2(x) = e^{3x}$ which is the same as the exact solution.

*Example 6.*Consider the nonlinear system of Volterra integro-differential equations

$$u_1' = 1 - x + \frac{x^2}{2} - \frac{x^4}{12} +$$
$$+\int_0^x \Big((x-t)u_1(t)^2 + u_2(t)^2\Big)dt, u_1(0) = 1$$
$$u_2' = -1 - x - \frac{3x^2}{2} - \frac{x^4}{12} +$$
$$\int_0^x \Big(u_1(t)^2 + (x-t)u_2(t)^2\Big)dt, u_2(0) = 1, \quad (22)$$

with the exact solution $u_1(x) = 1 + x$, $u_2(x) = 1 - x$. $u_0(x)$ is chosen to be

$$u_{10}(x) = \sum_{m=0}^{1} a_m x^m = a_0 + a_1 x$$
$$u_{20}(x) = \sum_{m=0}^{1} b_m x^m = b_0 + b_1 x$$

In view of Eq. (15) we have

$$u_{11}(x) = -a_0 - a_1x + 1 + x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^5}{60} +$$
$$+\int_0^x \Big(\int_0^x \Big((x-t)A_0[t] + B_0[t]\Big)dt\Big)ds = 0$$
$$u_{21}(x) = -b_0 - b_1x + 1 - x - \frac{x^2}{2} - \frac{x^3}{2} - \frac{x^5}{60} +$$
$$+\int_0^x \Big(\int_0^x \Big(A_0[t] + (x-t)B_0[t]\Big)dt\Big)ds = 0,$$

where $A_0$ and $B_0$ are the Adomian polynomials. Now $a_m$, $m = 0,1$ are found from the relation:

$$1 + x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{x^5}{60}$$
$$+\frac{1}{60}a_1^2x^5 + \frac{1}{4}\Big(\frac{1}{3}a_0a_1 + \frac{1}{3}b_1^2\Big)x^4$$
$$+\frac{1}{3}\Big(\frac{1}{2}a_0^2 + b_0b_1\Big)x^3$$
$$+\frac{1}{2}b_0^2x^2 - a_0 - a_1x = 0$$

as
$a_0 = 1$,
$a_1 = 1$.
Similarly, $b_m$, $m = 0,1$ are found from the relation:

$$1 - x - \frac{1}{2}x^2 - \frac{1}{2}x^3 - \frac{x^5}{60}$$
$$+\frac{1}{6}b_1^2x^5 + \frac{1}{4}\Big(\frac{1}{3}b_0b_1 + \frac{1}{3}a_1^2\Big)x^4$$
$$+\frac{1}{3}\Big(\frac{1}{2}b_0^2 + a_0a_1\Big)x^3 + \frac{1}{2}a_0^2x^2 - b_0 - b_1x = 0$$

as
$b_0 = 1$
$b_1 = -1$.
So, the solution will be $u_1(x) = 1 + x$ and $u_2(x) = 1 - x$ which is the same as the exact solution.

## 5 Conclusion

In this paper, the new modification of Adomian decomposition method is employed to solve linear and nonlinear integro-differential equation. It is evident from the numerical examples that the proposed method gives the exact solution after a single iteration. In this method the rate of convergence is accelerated. The method needs much less computational work compared with traditional methods. We recommend apply the method to other types of integral equations.

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