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Bayesian Prediction Based on General Progressive Censored Data from Generalized Pareto Distribution

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Abstract: This paper deals with the construct and compute in a Bayesian setting, point and interval predictions based on general progressive type II censored data from generalized Pareto distribution. Prediction bounds for the future observations (two sample prediction) based on this type of censored will be derived. Bayesian predictions are obtained based on a continuous–discrete joint prior for the unknown parameters. Bayesian point prediction under symmetric and asymmetric loss functions is discussed. As application, the total duration time in a life test and the failure time of a k-out-of-m system may be predicted. Finally, a real data set has been analyzed for illustrative purposes.

Keywords: Generalized Pareto distribution, General progressive type II censored, Bayesian predictions, Symmetric and asymmetric loss functions.

1 Introduction

A random variable X is said to have generalized Pareto (GP) distribution, if its probability density function (pdf) is given by

$$f_{(\zeta,\mu,\sigma)} = \frac{1}{\sigma}\left(1+\zeta\frac{x-\mu}{\sigma}\right)^{-(1/\zeta+1)}, \tag{1}$$

where $\mu,\zeta \in \mathbb{R}$ and $\sigma \in (0,+\infty)$. For convenience, we reparametrized this distribution by defining $\sigma/\zeta = \lambda, 1/\zeta = \alpha$ and $\mu = 0$. Therefore,

$$f(x) = \alpha\lambda(1+\lambda x)^{-(\alpha+1)}, \qquad x > 0. \tag{2}$$

The cumulative distribution function (cdf) and hazard function are given by

$$F(x) = 1-(1+\lambda x)^{-\alpha}, \qquad x > 0, \tag{3}$$

and

$$h(x) = \alpha\lambda(1+\lambda x)^{-1}, \qquad x > 0, \tag{4}$$

for $\alpha > 0$ and $\lambda > 0$. Here α and λ are the shape and scale parameters, respectively. It is also well known that this distribution has decreasing failure rate (DFR) property. This distribution is also known as Pareto distribution of the second type or Lomax distribution. [25] used generalized Pareto distribution to estimate the size of the maximum inclusion in clean steels and application of this distribution to reinsurance is discussed by [19]. The Pareto distribution of the second type has been widely used in economic studies and to analyse business failure data. The Pareto distribution has been studied by several authors. According to [4] the Pareto distribution is well adapted for modelling reliability problems, since many of its properties are interpretable in that context and could be an alternative to the well-known distributions used in reliability. This distribution was used for modelling size spectra data in aquatic ecology by [27]. [3] used the Pareto distribution as a mixing distribution for the Poisson parameter and obtained the discrete Poisson–Pareto distribution. [9] considered order statistics from non-identical right-truncated Lomax distributions and provided

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applications for this situation. [16] investigated the Bayesian estimation of the Pareto survival function. [15] determined the optimal times of changing stress level for simple stress plans under a cumulative exposure model using the Pareto distribution. [14] discussed the estimation of parameters of a Pareto distribution by generalized order statstics. [1] discussed some Bayesian inferences based on censored samples from the Pareto distribution. [10] considered progressively type II censored competing risks data from Pareto distributions.

In many life testing and reliability studies, the experimenter may not always obtain complete information on failure times for all experimental units. Data obtained from such experiments are called censored data. Saving the total time on test and the cost associated with it are some of the major reasons for censoring. Therefore, the test is considered to be censored in which data collected are the exact failure times on those failed units and the running times on those non-failed units. The most common censoring schemes are type-I and type-II censoring, but the conventional type-I and type-II censoring schemes do not have the flexibility of allowing removal of units at points other than the terminal point of the experiment. For example, some products have to be withdrawn for more thorough inspection or saved for use as test specimens in other studies. Different inferential procedures based on progressively censored samples have been discussed by several authors, including [5,6,7,12,21,23] and [11]. This paper considers a general progressive type II censoring scheme, this scheme can be described as follows: at time $X_0 = 0$, n randomly selected units were placed on a life test. The failure times of the first r units to fail, $X_1, \ldots, X_r$ were not observed. At the time of the $(r+1)th$ failure $X_{r+1:n}$, R_{r+1} number of surviving units are removed from the test randomly and so on. At the time of the $(r+i)th$ observed failure, $X_{r+i:n}$, R_{r+i} number of surviving units are removed from the test randomly. finally, at the time of the mth failure, the remaining $R_m = n - m - R_{r+1} - R_{r+2} - \cdots - R_{m-1}$ are removed from the test. Suppose that $X_{r+1:m:n} \leq X_{r+2:m:n} \leq \cdots \leq, X_{m:m:n}$ are the lifetimes of the completely observed units to fail and that $R_{r+1}, R_{r+2}, \ldots, R_m$ are the number of units removed from the test at these failure times, respectively. The R_i's, m, and r are prespecified integers such that $0 \leq r < m \leq n$, $0 \leq R_i \leq n - i$ for $i = r+1, \ldots, m-1$, and $R_m = n - m - \sum_{i=r+1}^{m-1} R_i$. The resulting $(m-r)$ ordered values $X_{r+1:m:n}, X_{r+2:m:n}, \ldots, X_{m:m:n}$ are appropriately referred to as general progressively type II censored order statistics, it should be noted that

(i)If $r = 0$, this scheme is reduced to the progressive type II right censoring.
(ii)If $r = 0$ and $R_i = 0$, for $i = r+1, \ldots, m-1$, then $R_m = n - m$, the general progressively type II censoring scheme is reduced to conventional type II one-stage right censoring, where just the first m usual order statistics are observed.
(iii)If $r = 0$ and $R_i = 0$, for $i = r+1, \ldots, m$, then $m = n$, the general progressively type II censoring scheme is reduced to complete sample case (the case of no censoring), where all n usual order statistics are observed.
(iv)If $r \neq 0$ and $R_i = 0$, for $i = r+1, \ldots, m-1$, then $R_m = n - m - r$, the general progressively type II censoring scheme is reduced to the doubly type II censoring scheme.

[18] discussed the estimation problem for Rayleigh distribution based on a general progressive censored schemes. [17] investigated the estimating Burr XII parameter based on general type II progressive censoring. [24] discussed the problem of Bayesian estimation and prediction based on multiply type II censored samples of sequential order statistics from one and two-parameter exponential distributions. This paper focuses, via Bayesian approach, on two-sample predictive inferences for the generalized Pareto distribution based on a general progressively type II censored data.

The layout of the paper is as follows: The likelihood function, prior and posterior distributions are presented in Section 2. Section 3 presents the different types of the loss functions. The details of our main results along with the derivation of all Bayes predictive functions based on general progressive censored data are given in Section 4. In the same section, the predictive functions were used to derive both point prediction and prediction intervals for the future observations from the same distribution. Numerical example using real data set is presented in Section 5. Finally, Section 6 provides some concluding remarks.

2 The Likelihood Function and Posterior Distribution

Let $\mathrm{X} = X_{r+1:m:n}, X_{r+2:m:n}, \ldots, X_{m:m:n}$ denote the general progressively type II censored sample from the generalized Pareto distribution, with $(R_{r+1}, R_{r+2}, \ldots, R_m)$ being the general progressive censoring scheme, and r is the number of the first failures which are not observed. For simplicity of notation, we use x_i instead of $X_{i:m:n}$ with $i = r+1, \ldots, m$. The likelihood function $\ell(\alpha, \lambda | \mathbf{x})$ for the parameters α and λ is then

$$\ell(\alpha, \lambda | \mathbf{x}) = al \; \{F(x_{r+1})\}^r \prod_{i=r+1}^{m} f(x_i) \left[1 - F(x_i)\right]^{R_i}, \tag{5}$$

where

$$al = \binom{n}{r} (n-r) \prod_{j=r+2}^{m} \left[n - \sum_{i=r+1}^{j-1} R_i - j + 1 \right], \tag{6}$$

and the functions $f(x)$ and $F(x)$ are given respectively by (2) and (3). Substituting (2) and (3) into (5) the likelihood function is

$$\ell(\alpha,\lambda|\mathbf{x}) = al\ \left[1-(1+\lambda x_{r+1})^{-\alpha}\right]^r \prod_{i=r+1}^{m} \alpha\lambda\,(1+\lambda x_i)^{-\alpha(R_i+1)-1}. \tag{7}$$

By using the binomial expansion where r is a positive integer, one can rewrite the likelihood function (7) as follows

$$\ell(\alpha,\lambda|\mathbf{x}) = al \sum_{s=0}^{r} (-1)^s \binom{r}{s} (\alpha\lambda)^{m-r} u e^{-\alpha T_s}, \tag{8}$$

where

$$u = \prod_{i=r+1}^{m} (1+\lambda x_i)^{-1}, \quad T_s = s\ln(1+\lambda x_{r+1}) + \sum_{i=r+1}^{m} (R_i+1)\ln(1+\lambda x_i). \tag{9}$$

Now, we first describe the prior information needed for the Bayesian analysis of the unknown parameters. When the parameters λ and α, are assumed to be unknown, we assume that the parameter λ has a discrete prior distribution, while α has a conjugate prior distribution. Suppose that the parameter λ is restricted to a finite number of values, say $\lambda_1,\lambda_2,...,\lambda_N$, with prior probabilities $\eta_1,\eta_2,...,\eta_N$, respectively, where $0\le\eta_j\le 1$ and $\sum_{j=1}^{N}\eta_j = 1$. i.e.

$$\pi(\lambda_j) = \Pr(\lambda=\lambda_j) = \eta_j. \tag{10}$$

Under the condition $\lambda=\lambda_j,\ j=1,2,...,N$, suppose that α has a natural conjugate gamma prior with parameters a_j and b_j

$$\pi(\alpha|\lambda=\lambda_j) = \frac{b_j^{a_j}}{\Gamma(a_j)}\alpha^{a_j-1}e^{-b_j\alpha}, \quad \alpha;\ a_j,\ b_j>0. \tag{11}$$

Using the likelihood function (8) and the prior density (11), the conditional posterior density of α given $\lambda=\lambda_j$ is

$$\pi^*(\alpha|\lambda=\lambda_j;\mathbf{x}) = k_1 \sum_{s=0}^{r} (-1)^s \binom{r}{s} \alpha^{A_j-1}e^{-\alpha B_j}, \quad \alpha>0, \tag{12}$$

where

$$A_j = m-r+a_j, \quad B_j = T_s+b_j, \quad k_1^{-1} = \sum_{s=0}^{r}(-1)^s\binom{r}{s}\Gamma(A_j)B_j^{-A_j}. \tag{13}$$

The joint posterior of α and λ is

$$\pi^*(\alpha,\lambda|\mathbf{x}) = k_2 \sum_{s=0}^{r}(-1)^s\binom{r}{s}D_j\alpha^{A_j-1}e^{-\alpha B_j}, \tag{14}$$

where

$$\begin{aligned} D_j &= \frac{b_j^{a_j}}{\Gamma(a_j)}\eta_j\lambda^{m-r}u_j, \quad u_j = \prod_{i=r+1}^{m}(1+\lambda_j x_i)^{-1}, \\ k_2^{-1} &= \sum_{j=1}^{N}\sum_{s=0}^{r}(-1)^s\binom{r}{s}D_j\Gamma(A_j)B_j^{-A_j} \end{aligned} \tag{15}$$

3 The Loss Functions

For Bayesian approach, in order to select a single value as representing our “best” estimator of the unknown parameter, a loss function must be specified. A wide variety of loss functions have been developed in literature to describe various types of loss structures. The symmetric square-error loss (SE) is one of the most popular loss functions. It is widely employed in inference, but its application is motivated by its good mathematical properties, not by its applicability to representing a true loss structure. A loss function should represent the consequences of different errors. There are situations where over- and under-estimation can lead to different consequences. For example, when we estimate the average reliable working life of the components of a spaceship or an aircraft, over-estimation is usually more serious than under-estimation than an underestimation. Being symmetric, the SE loss equally penalizes over- and under-estimation of the same magnitude.

A useful asymmetric loss known as the LINEX loss function, was introduced by [26]. This function rises approximately exponentially on one side of zero, and approximately linearly on the other side. Under the assumption that the minimal loss occurs at $\widetilde{u}=u$, the LINEX loss function for u can be expressed as

$$L(\Delta) \propto e^{c\Delta} - c\Delta - 1,\ c \neq 0, \tag{16}$$

where $\Delta = (\widetilde{u}-u)$ and $\widetilde{u}$ is an estimate of u .
It is easy to verify that the value of $\widetilde{u}$ that minimizes $E_u\left(L(\widetilde{u}-u)\right)$ in (16) is

$$\widetilde{u}_{BL} = -\frac{1}{c}\log(E_u[\exp(-cu)]). \tag{17}$$

Another useful asymmetric loss function is the general entropy (GE) loss

$$L_1(\widetilde{u},u) \propto \left(\frac{\widetilde{u}}{u}\right)^q - q\log\left(\frac{\widetilde{u}}{u}\right) - 1. \tag{18}$$

whose minimum occurs at $\widetilde{u}=u$. The Bayes predictive estimate $\widetilde{u}_{BL}$ of u under GE loss (18) is

$$\widetilde{u}_{BG} = (E_u[u^{-q}])^{-1/q}. \tag{19}$$

For more details about these loss functions see [2] and [13].

4 Bayesian Prediction

Suppose that $X_{r+1}, X_{r+2}, \ldots, X_m$ is general progressively type II censored sample drawn from a population whose density function is $\mathrm{GP}(\alpha,\lambda)$ defined in (2), and that $Y_1, Y_2, \ldots, Y_{n_1}$ is a second independent random sample of size n_1 of the future observations from the same distribution. It is further assumed that the two samples are independent and each of their corresponding random samples is obtained from the same distribution function. Our aim is to develop a method to construct a prediction interval for a number of future observations (two-sample prediction). Let Y_k $(1 \leq k \leq n_1)$, be the k^{th} ordered lifetime in the future sample of size n_1. The density function of Y_k for given α and λ is

$$g(y_k|\alpha,\lambda) = k\binom{n_1}{k}\left[1-F\left(y_k|\alpha,\lambda\right)\right]^{n_1-k}\left[F\left(y_k|\alpha,\lambda\right)\right]^{k-1} f\left(y_k|\alpha,\lambda\right),\ \alpha,\lambda > 0, \tag{20}$$

where $f\left(.|\alpha,\lambda\right)$ is given in (2) and $F\left(.|\alpha,\lambda\right)$ denotes the corresponding cumulative distribution function of $f\left(.|\alpha,\lambda\right)$ as given in (3), substituting (2) and (3) in (20),we obtain

$$g(y_k|\alpha,\lambda) = G\left(k\right)\sum_{i=0}^{k-1}(-1)^i\binom{k-1}{i}\left(1+\lambda y_k\right)^{-1}\alpha\lambda e^{-\alpha n^*\ln(1+\lambda y_k)}, \tag{21}$$

where

$$G\left(k\right) = k\binom{n_1}{k}, \quad n^* = n_1 - k + i + 1. \tag{22}$$

Bayes predictive density function of Y_k given $\mathbf{x}$ is

$$g_1(y_k|\mathbf{x}) = \int_0^\infty \sum_{j=1}^{N} g(y_k|\alpha,\lambda)\pi^*(\alpha,\lambda|\mathbf{x})d\alpha, \tag{23}$$

where $\pi^*(\alpha,\lambda|\mathbf{x})$ is the joint posterior density of α and λ as given in (14). Substituting (14) and (21) in (23), Bayes predictive density function of Y_k can be written as

$$g_1(y_k|\mathbf{x}) = \sum_{j,i,s}^{N,k-1,r} k_2 G\left(k\right)\lambda_j D_j\left(1+\lambda_j y_k\right)^{-1}\frac{\Gamma\left(A_j+1\right)}{\left[B_j + n^*\ln\left(1+\lambda_j y_k\right)\right]^{\left(A_j+1\right)}}, \tag{24}$$

where

$$\sum_{j,i,s}^{N,k-1,r} = \sum_{j=1}^{N}\sum_{i=0}^{k-1}\sum_{s=0}^{r}(-1)^{i+s}\binom{k-1}{i}\binom{r}{s}. \tag{25}$$

Thus the predictive survival function $Pr[Y_k \geq t|\mathbf{x}]$ can be written as

$$\begin{aligned} \Pr[Y_k \geq t|\mathbf{x}] &= \int_t^\infty g_1(y_k|\mathbf{x})dy_k, \\ &= \sum_{j,i,s}^{N,k-1,r} k_2 G(k) D_j \frac{\Gamma(A_j+1)}{n^* A_j [B_j + n^* \ln(1+\lambda_j t)]^{A_j}}. \end{aligned} \tag{26}$$

4.1 Interval prediction

The predictive bounds of a two-sided interval with cover γ, for the future observation Y_k, can be obtained by solving the following two equations for lower L and upper U bounds

$$\Pr[Y_k \geq L|\mathbf{x}] = \frac{1+\gamma}{2}, \qquad \Pr[Y_k \geq U|\mathbf{x}] = \frac{1-\gamma}{2} \tag{27}$$

Special cases:

(i)Setting $k=1$ in (26), yields

$$\Pr[Y_1 \geq t|\mathbf{x}] = \sum_j^N \sum_s^r (-1)^s \binom{r}{s} k_2 D_j \frac{\Gamma(A_j+1)}{A_j [B_j + n_1 \ln(1+\lambda_j t)]^{A_j}}. \tag{28}$$

This case is of particular interest; for instance, a lower limit for the first failure in a fleet of n_1 items is called a safe warranty life or an assurance limit for the fleet.

(ii)Setting $k=n_1$ in (26), yields

$$\Pr[Y_{n_1} \geq t|\mathbf{x}] = \sum_{j,i,s}^{N,n_1-1,r} k_2 D_j \frac{\Gamma(A_j+1)}{(i+1) A_j [B_j + (i+1)\ln(1+\lambda_j t)]^{A_j}}. \tag{29}$$

A 100γ Bayesian prediction interval for Y_1 and Y_{n_1} can be easily obtained numerically using (27)–(29).

4.2 Point prediction

Using (17), (19) and (24) Bayes point predictor of Y_k under LINEX and GE loss functions are given, respectively, by

$$\begin{aligned} \tilde{Y}_{k_{(BL)}} &= -\frac{1}{c} Log\left[\int_0^\infty e^{-cy_k} g_1(y_k|\mathbf{x}) dy_k\right] \\ &= -\frac{1}{c} Log\left[\sum_{j,i,s}^{N,k-1,r} k_2 G(k) D_j \lambda_j \Gamma(A_j+1) I_1(n^*)\right], \end{aligned} \tag{30}$$

$$\begin{aligned} \tilde{Y}_{k_{(BG)}} &= \left[\int_0^\infty y_k^{-q} g_1(y_k|\mathbf{x}) dy_k\right]^{\frac{-1}{q}} \\ &= \left[\sum_{j,i,s}^{N,k-1,r} k_2 G(k) D_j \lambda_j \Gamma(A_j+1) I_2(n^*,k)\right]^{\frac{-1}{q}}, \end{aligned} \tag{31}$$

where

$$\begin{aligned} I_1(n^*) &= \int_0^\infty \frac{e^{-cy_k}\ (1+\lambda_j y_k)^{-1}}{[B_j + n^* \ln(1+\lambda_j y_k)]^{(A_j+1)}} dy_k, \\ I_2(n^*,k) &= \int_0^\infty \frac{y_k^{-q}\ (1+\lambda_j y_k)^{-1}}{[B_j + n^* \ln(1+\lambda_j y_k)]^{(A_j+1)}} dy_k. \end{aligned} \tag{32}$$

One can use a numerical integration technique to compute the integrals in (32). It may be noted that the Bayes point predictor $\tilde{Y}_{k_{(BS)}}$ for y_k under squared error loss can be obtained by setting $q=-1$ in (31).

Special cases:

(i)When $k=1$, the point prediction will be for the first failure time of the future sample of size n_1. Setting $k=1$ in (30) and (31), respectively, we obtain

$$\tilde{Y}_{1_{(BL)}} = -\frac{1}{c}Log\left[\sum_{j,i,s}^{N,0,r} n_1 k_2 D_j \lambda_j \Gamma\left(A_j+1\right) I_1\left(n_1\right)\right], \tag{33}$$

$$\tilde{Y}_{1_{(BG)}} = \left[\sum_{j,i,s}^{N,0,r} n_1 k_2 D_j \lambda_j \Gamma\left(A_j+1\right) I_2\left(n_1,1\right)\right]^{\frac{-1}{q}}. \tag{34}$$

(ii)When $k=n_1$, the point prediction will be for the last failure time of the future sample of size n_1. Setting $k=n_1$ in (30) and (31), respectively, gives

$$\tilde{Y}_{n_{1(BL)}} = -\frac{1}{c}Log\left[\sum_{j,i,s}^{N,n_1-1,r} n_1 k_2 D_j \lambda_j \Gamma\left(A_j+1\right) I_1\left(i+1\right)\right], \tag{35}$$

$$\tilde{Y}_{n_{1(BG)}} = \left[\sum_{j,i,s}^{N,n_1-1,r} n_1 k_2 D_j \lambda_j \Gamma\left(A_j+1\right) I_2\left(i+1,n_1\right)\right]^{\frac{-1}{q}}. \tag{36}$$

5 Numerical Computations

To illustrate the application of the prediction results to the analysis of survival data, we consider the data set in [29] which was analyzed also in [20]. These data are the duration of remission of 20 leukemia patients which are treated by one drug. The ordered duration of remission (in years) are:

1.013	1.034	1.109	1.169	1.226	1.509	1.533	1.563	1.716	1.929
1.965	2.061	2.344	2.546	2.626	2.778	2.951	3.413	4.118	5.136

Before progressing further, the 20 values were used to verify that the data set follow generalized Pareto distribution GP(α,λ), We have examined the goodness of fit of the previous data to generalized Pareto distribution. We have computed the Kolmogorov-Smirnov test. it is 0.0942 and the corresponding p-value is 0.886. Since the p-value is quite high, we cannot reject the null hypothesis that the data is coming from the generalized Pareto distribution.

In this example, the following 6 censoring schemes (C.S) are considered:

(1) General progressive type II censored sample ($r=1$, $n=20$, $m=9$) can be generated by algorithm given by [8]. In that sample, the first time to breakdown is lost and the vector of observed failure times and the progressive censoring scheme are given by

$\mathbf{x}=(\square,\ 1.034,\ 1.109,\ 1.169,\ 1.533,\ 1.563,\ 2.061,\ 2.344,\ 2.546)$
and $R_i=(0,\ 2,\ 0,\ 0,\ 3,\ 0,\ 0,\ 6)$, $i=r+1,...,m$.

(2) Usual progressive type II censored sample ($r=0$, $n=20$, $m=9$) generated by [28]. In that sample the vector of observed failure times and the progressive censoring scheme are

$\mathbf{x}=(1.013,\ 1.034,\ 1.109,\ 1.169,\ 1.533,\ 1.563,\ 2.061,\ 2.344,\ 2.546)$
and $R_i=(0,\ 0,\ 2,\ 0,\ 0,\ 3,\ 0,\ 0,\ 6)$, $i=r+1,...,m$.

(3) Another usual progressive type II censored sample ($r=0$, $n=20$, $m=9$) generated by [22] using the optimal censoring plan. The observed failure times and the progressive censoring scheme are

$\mathbf{x}=(1.013,\ 1.034,\ 1.169,\ 2.061,\ 2.546,\ 2.778,\ 2.951,\ 3.413,\ 4.118)$
and $R_i=(0,\ 11,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0)$, $i=r+1,...,m$.

(4) Doubly type II censored sample ($r=4$, $n=20$, $m=14$). In this case we suppose that 20 specimens were put on a test, but for reasons of economy it was decided to terminate the test on the 14-th failure. Moreover, due to experimental difficulties, the failure times of the first four specimens to fail were missing. The ten observed failure times and the progressive censoring scheme are as follows

$\mathbf{x}=(\square,\square,\square,\square,1.266,1.509,1.533,1.563,1.716,1.929,1.965,2.061,2.344,2.546,\square,\square,\square,\square,\square,\square)$
and $R_i=(0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 6)$, $i=r+1,...,m$.

(5) Usually type II censored sample ($r=0$, $n=20$, $m=9$).

$\mathbf{x}=(1.013,\ 1.034,\ 1.109,\ 1.169,\ 1.266,\ 1.716,\ 1.929,\ 1.965,\ 2.061)$
and $R_i=(0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 11)$, $i=r+1,...,m$.

(6) Complete sample ($r=0$, $n=m=20$).

$\mathbf{x} = (1.013, 1.034, 1.109, 1.169, 1.266, 1.509, 1.533, 1.563, 1.716, 1.929, 1.965, 2.061, 2.344, 2.546, 2.626, 2.778, 2.951, 3.413, 4.118, 5.136)$ and $R_i = (0^{20})$, $i = r+1, ..., m$.
The values of the parameters a_j and b_j are obtained numerically for each given λ_j, and η_j, $j = 1, 2, ..., 10$ using the Newton–Raphson method. Table 1 summarized the values of a_j and b_j for each given λ_j and η_j. The estimation of the parameters c and q would require considerable information about the true losses for the producer. However, for purposes of this study, we choose some different values for c and q. Now assume that thirteen ($n_1 = 13$) new insulation specimens are to be tested. On the basis of the preceding sets of data $\mathbf{x}$, Bayes point prediction, under SE, LINEX, and GE loss functions, of the future failure times Y_1, Y_{13} and the corresponding 95 Bayes prediction intervals are shown in Tables 2 and 3.

Table 1: Prior information and hyper parameter values.

j	1	2	3	4	5	6	7	8	9	10
η_j	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
λ_j	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
a_j	0.6992	0.5965	0.5183	0.4570	0.4077	0.3674	0.3339	0.3056	0.2814	0.2605
b_j	1.4646	1.3745	1.3034	1.2456	1.1977	1.1573	1.1227	1.0928	1.0665	1.0433

Table 2: Point prediction and 95 prediction interval for Y_1 .

C.S	Point Prediction					Interval Prediction	
	BS	BL		BG		LL	UL
		$c = 0.5$	$c = 1$	$q = -0.6$	$q = 0.9$		
(1)	1.3245	1.3123	1.2889	1.2544	1.2458	0.0615	3.2415
(2)	1.3056	1.2945	1.2837	1.2456	1.2366	0.0422	3.1923
(3)	1.3333	1.2921	1.2754	1.2369	1.2310	0.0343	3.7169
(4)	1.3161	1.2544	1.2733	1.2354	1.2155	0.0406	3.2111
(5)	1.4002	1.3923	1.2824	1.2844	1.3041	0.0397	3.6251
(6)	1.3770	1.3642	1.3122	1.2937	1.2862	0.0238	3.2477

Table 3: Point prediction and 95 prediction interval for Y_{13}.

C.S	Point Prediction					Interval Prediction	
	BS	BL		BG		LL	UL
		$c = 0.01$	$c = 0.5$	$q = -0.1$	$q = 0.4$		
(1)	37.1245	31.1245	33.4358	37.3341	31.0840	6.9547	88.8415
(2)	39.5521	31.6540	32.7451	36.5487	31.6652	9.2354	87.3642
(3)	42.3258	41.2547	45.6984	41.1124	40.1347	12.1139	99.3210
(4)	32.2261	28.2457	33.8686	31.5479	27.9687	12.6982	102.8547
(5)	47.2587	50.0025	48.2105	55.6478	51.5784	14.5421	111.5694
(6)	38.1458	41.1147	42.6391	44.0124	39.1502	13.6317	95.8563

6 Conclusions

In this paper, Bayesian prediction problem of the generalized Pareto distribution based on general progressively censored sampling are obtained. The prior belief of the model is represented by a continuous-discrete joint prior for the unknown parameters. Using the predictive distribution approach, the methods are derived for constructing either point and interval predictions for the kth smallest future observations from the same failure process. All of the results obtained in this paper can be specialized to: usual progressive type II censored sample, type II right censored sample, doubly type II censored sample and complete sample. Bayesian approach provides a unified structure for computing prediction intervals to one out of m, k out of m, and m out of m sample values with a desired probability. It also, provides a convenient computational setting for actual calculations of the prediction intervals. Such calculations can be accomplished easily using widely available computing facilities. Bayesian approach allows prior knowledge as well as experimental data to be incorporated into the inferential procedure, thus avoiding the well known difficulties of the classical approach in analyzing censored data. The application examples presented in this paper illustrate the procedure of using a real general progressively censored data as a past sample to predict future samples from the same population. Bayes point prediction under asymmetric loss (LINEX and GE) is more general, which includes point prediction based on symmetric loss functions as a special cases. So, the analytical ease with which results can be obtained using asymmetric loss functions makes them attractive for using in applied problems, and in assessing the effects of departures from assumed symmetric loss functions. From Tables 2 and 3 Bayes point prediction relative to asymmetric loss functions are sensitive to the value of the parameters c and q. These results establish that for optimum decision making, importance should be given to the choice of loss function and not just the choice of prior distribution only.

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