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Global Error for Generalized Complementarity Problems based on Generalized Fisher-Burmeister Function

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Abstract: In this paper, we provide global error bounds generalized complementarity problem, denoted by GCP(f,g) based on the generalized Fisher-Burmeister function and its generalizations under certain conditions. These error bounds can be viewed not only as extensions of previously known results the existing but also new results for the nonlinear/generalized complementarity problems.

Keywords: Error bounds, generalized complementarity problem, GCP function, generalized FB function, natural residual.

1 Introduction

It is well known that error bounds can be used to measure how much the approximate solution fails to be in the solution set and to analyze the convergence rates of various methods. The study of the error bounds has received increasing attention in the last decades, see the survey paper by Pang [23] and references therein. The error bounds have played important roles in stopping rules and the convergence analysis for many iterative algorithms, and in the treatment of various issues in the areas of complementarity problems and variational inequalities. Further, it can be used in the sensitivity analysis of the problems when their data is subject to perturbation [6], [9], [23].

The global error bound, that is, an upper bound estimation of the distance from a given point in R^n to the solution set of the problem in terms of some residual functions, is an important one [23]. The error bound estimation for the classical linear complementarity problems (LCP) [17], [18], [20], [30], nonlinear complementarity problems [19] and the generalized linear complementarity problems over a polyhedral cone [26] were analyzed. In the special cases of linear/nonlinear complementarity problems, the natural residual has played an important role in establishing global error bounds, see, e.g., [17], [19], [20]. It plays a similar role in our analysis for generalized nonlinear complementarity problems [16]. In this paper, we study the global error for generalized nonlinear complementarity problems based on the generalized Fisher-Burmeister function and its generalizations under certain conditions.

1.1 The Problem

Consider the generalized complementarity problem corresponding to f and g, denoted by GCP(f,g), which is to find a vector $x^* \in \Re^n$ such that

$$f(x^*) \geq 0, \quad g(x^*) \geq 0 \quad \text{and} \quad \langle f(x^*), g(x^*) \rangle = 0 \qquad (1)$$

where $f : \Re^n \to \Re^n$ and $g : \Re^n \to \Re^n$ are given C^1 functions.

For the numerical methods formulation, and applications of GCP(f,g), we refer the interested readers to [14], [15], [22] and the references cited therein. Also GCP(f,g) is a generalization of the nonlinear complementarity problem NCP(f) when $g(x) = x$, linear complementarity problem LCP(M,q) when $f(x) = Mx + q$ and $g(x) = x$ with $M \in R^{n\times n}$ and a vector $q \in R^n$, and quasi/implicit complementarity problem when $g(x) = x - W(x)$ with some $W : R^n \to R^n$, see, e.g., [15], [21], [24].

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The importance of these problems in operations research, optimization, engineering sciences, economics and other areas has been well documented in the literature, see e.g., [4], [5], [7], [8], [13], [25], and the references therein.

One of the popular methods for solving GCP(f,g) is to reformulate it as a minimization problem, see, e.g., [11], [16], [28]. A function which can constitute an equivalent minimization problem for the GCP(f,g) is called a merit function. More specifically, a merit function is a function whose global minima on a set $X \in R^n$ are coincident with the solutions of the original GCP(f,g). In order to construct thi merit function, we need to define GCP functions. A function $\phi : R^2 \to R$ is called a GCP function if

$$\phi(a,b) = 0 \Leftrightarrow ab = 0, a \geq 0, b \geq 0.$$

We call

$$\Phi(x) = \begin{bmatrix} \phi(f_1(x), g_1(x)) \\ \vdots \\ \phi(f_i(x), g_i(x)) \\ \vdots \\ \phi(f_n(x), g_n(x)) \end{bmatrix} \tag{2}$$

a GCP function for GCP(f,g).

We consider a GCP function $\Phi : R^n \to R^n$ associated with GCP(f,g) and its merit function

$$\Psi_*(\bar{x}) := \frac{1}{2}\|\Phi_*(\bar{x})\|^2 = \sum_{i=1}^{n} \psi_*(f_i(\bar{x}), g_i(\bar{x})), \tag{3}$$

where $\Phi_*(\bar{x})$ is defined in (2) and

$$\psi_*(a,b) := \frac{1}{2}\phi_*(a,b)^2, \tag{4}$$

with $* \in \{\{1,p\},1,2,3,4,\{\theta,p\}\}$.

And for $\psi_{\alpha,\theta,p}$, we denote the corresponding merit function as

$$\Psi_{\alpha,\theta,p}(\bar{x}) := \sum_{i=1}^{n} \phi_{\alpha,\theta,p}(f_i(\bar{x}), g_i(\bar{x})). \tag{5}$$

So that

$$\Psi_*(\bar{x}) = 0 \Leftrightarrow \Phi_*(\bar{x}) = 0 \Leftrightarrow \bar{x} \text{ solves GCP}(f,g).$$

If we assume GCP(f,g) has at least one solution, then a vector $\bar{x} \in R^n$ solves GCP(f,g) if and only if it is a global/local minimizer (a stationary point) of the unconstrained minimization problem

$$\min_{x \in R^n} \Psi(x).$$

1.2 Example of GCP functions

We give some examples for GCP functions based on the generalized Fisher-Burmeister function and it generalization [2,1,12]. Suppose that f and g are C^1.

Example 1 Consider the following GCP function which is the basis of

$$\phi_p(a,b) := a + b - \|(a,b)\|_p \tag{6}$$

where p is any fixed real number in the interval $(1,+\infty)$ and $\|(a,b)\|_p$ denotes the p-norm of (a,b), i.e., $\|(a,b)\|_p = \sqrt[p]{|a|^p + |b|^p}$. The function ϕ_p was noted by Tseng [29]. For further study on this family of NCP functions, see [2].

The ith component of GCP function $\Phi(x)$ in (2) is defined as

$$\Phi_i(x) = \phi_p(f_i(x), g_i(x)) := f_i(x) + g_i(x) - \|(f_i(x), g_i(x))\|_p$$

Example 2 Consider the following GCP function which is based on proposed family of NCP functions [2] relying on ϕ_p in (6) and some introduced NCP functions in [1]:

$$\phi_1(a,b) := \phi_p(a,b) + \alpha a_+ b_+, \quad \alpha > 0. \tag{7}$$

The ith component of GCP function $\Phi(x)$ in (2) is defined as

$$\begin{aligned} \Phi_i(x) &= \phi_1(f_i(x), g_i(x)) \\ &:= \phi_p(f_i(x), g_i(x)) + \alpha f_i(x)_+ g_i(x)_+, \quad \alpha > 0. \end{aligned}$$

Example 3 The following GCP function is based on NCP function in [2]

$$\phi_2(a,b) := \phi_p(a,b) + \alpha(ab)_+, \quad \alpha > 0. \tag{8}$$

We define the ith component of GCP function $\Phi(x)$ in (2) as

$$\begin{aligned} \Phi_i(x) &= \phi_2(f_i(x), g_i(x)) \\ &:= \phi_p(f_i(x), g_i(x)) + \alpha(f_i(x) g_i(x))_+, \quad \alpha > 0. \end{aligned}$$

Example 4 The following GCP function is based on NCP function in [2]

$$\phi_3(a,b) := \sqrt{[\phi_p(a,b)]^2 + \alpha(a_+ b_+)^2}, \quad \alpha > 0. \tag{9}$$

We define the ith component of GCP function $\Phi(x)$ in (2) as

$$\begin{aligned} \Phi_i(x) &= \phi_3(f_i(x), g_i(x)) \\ &:= \sqrt{[\phi_p(f_i(x), g_i(x))]^2 + \alpha(f_i(x)_+ g_i(x)_+)^2}, \quad \alpha > 0 \end{aligned}$$

Example 5 Consider following GCP function

$$\phi_4(a,b) := \sqrt{[\phi_p(a,b)]^2 + \alpha[(ab)_+]^2}, \quad \alpha > 0. \tag{10}$$

This function is motivated from NCP function in [2]. Define the ith component of GCP function $\Phi(x)$ in (2) as

$$\begin{aligned} \Phi_i(x) &= \phi_3(f_i(x), g_i(x)) \\ &:= \sqrt{[\phi_p(f_i(x), g_i(x))]^2 + \alpha[(f_i(x) g_i(x))_+]^2} \end{aligned}$$

where $\alpha > 0$.

Example 6 We consider the following GCP function which is based on another family of NCP functions [12]

$$\phi_{\theta,p}(a,b) := a+b-\sqrt[p]{\theta(|a|^p+|b|^p)+(1-\theta)|a-b|^p} \quad (11)$$

where $\theta \in (0,1]$. It is clear that (11) reduce to (6) when $\theta = 1$. In the following, we will denote

$$\phi_{1,p}(a,b) = \phi_p(a,b) = a+b-\|(a,b)\|_p.$$

We define the ith component of GCP function $\Phi(x)$ in (2) as

$$\Phi_i(x) = \phi_{\theta,p}(f_i(x),g_i(x)) := f_i(x)+g_i(x) \\ -\sqrt[p]{\theta(|f_i(x)|^p+|g_i(x)|^p)+(1-\theta)|f_i(x)-g_i(x)|^p}$$

where $\theta \in (0,1]$.

Example 7 Based on (11) and NCP function in [2], we consider the following GCP function

$$\phi_{\alpha,\theta,p}(a,b) := \frac{\alpha}{2}[(ab)_+]^2+\frac{1}{2}[\phi_{\theta,p}(a,b)]^2, \quad \alpha \geq 0 \quad (12)$$

where $\phi_{\alpha,\theta,p}(a,b) : R^2 \to R_+$. The ith component of GCP function $\Phi(x)$ in (2) is defined as

$$\phi_{\alpha,\theta,p}(f_i(x),g_i(x)) \\ := \frac{\alpha}{2}[(f_i(x)g_i(x))_+]^2+\frac{1}{2}[\phi_{\theta,p}(f_i(x),g_i(x))]^2, \quad \alpha \geq 0.$$

In this article, we give global error bounds generalized complementarity problem, denoted by GCP(f,g) based on the generalized Fisher-Burmeister function and its generalizations under Relatively uniform **(P)**- conditions. These error bounds can be viewed as extensions of previously known results in [17], [20], [19] [16].

A word about our notation. Vector inequalities are interpreted componentwise. Vectors in R^n are regarded as column vectors. The inner-product between two vectors x and y in R^n is denoted by either $x^T y$ or $\langle x,y\rangle$. For a matrix A, the ith row of A is denoted by A_i. For a differentiable function $f : R^n \to R^m$, the Jacobian matrix of f at $\bar{x}$ is denoted by $\nabla f(\bar{x})$. The p-norm of x is denoted $||x||_p$ and the Euclidean norm of x is denoted by $||x||$. Assume that p is a fixed real number in $(1,\infty)$. Also, we use the natural residual merit function $\Psi_{NR} : R^n \to R_+$ defined by $\Psi_{NR}(x) := \frac{1}{2}\sum_{i=1}^{n}\phi_{NR}^2(f_i(x),g_i(x))$ where $\phi_{NR} : R^2 \to R$ denotes the minimum GCP-function $\min\{a,b\}$.

In [27], the author generalized the concepts of monotonicity, $\mathbf{P_0}$-property and their variants for functions and use them to establish some conditions to get a solution for generalized complementarity problem when the underlying functions f and g are H-differentiable. .

Let us recall the following definitions from [27].

Definition 1 *For functions $f,g : \Re^n \to \Re^n$, we say that f and g are:*

(a) Relatively monotone if

$$\langle f(x)-f(y), g(x)-g(y)\rangle \geq 0 \quad \text{for all } x,y \in \Re^n.$$

(b) Relatively strictly monotone if

$$\langle f(x)-f(y), g(x)-g(y)\rangle > 0 \quad \text{for all } x,y \in \Re^n.$$

(c) Relatively strongly monotone if there exists a constant $\mu > 0$ such that

$$\langle f(x)-f(y), g(x)-g(y)\rangle \geq \mu\|x-y\|^2 \quad \text{for all } x,y \in \Re^n.$$

(d) Relatively $\mathbf{P_0}(\mathbf{P})$-functions if for any $x \neq y$ in $\Re^n$,

$$\max_{i:x_i\neq y_i}[f(x)-f(y)]_i[g(x)-g(y)]_i \geq (>)0.$$

*(e) Relatively uniform **(P)**-functions if there exists a constant $\eta > 0$ such that for any $x,y \in \Re^n$,*

$$\max_{1\leq i\leq n}[f(x)-f(y)]_i[g(x)-g(y)]_i \geq \eta\|x-y\|^2.$$

Note that relatively strongly monotone functions are relatively strictly monotone, and relatively strictly monotone functions are relatively monotone.

2 The main result

We start by defining the concepts of a residual, lower and upper error bounds.

Definition 2 *Let $e : \Re^n \to \Re, X \in\subseteq \Re^n$ and let*

$$dist(x,X) := \inf_{y\in X}\|x-y\|$$

denote the distance of an arbitrary vector x to the set X. Assume that GCP(f,g) has a nonempty solution set X^*. Then:

(a) the function e is called a residual of GCP(f,g) if $e(x) \geq 0$, for all $x \in \Re^n$, and $e(x) = 0$ if and only if x solves GCP(f,g);

(b) a residual e is lower global error bound for GCP(f,g) if there exists some constant $\tau_1 > 0$ such that $\tau_1 e(x) \leq dist(x,X^*)$, for each $x \in \Re^n$; a residual e is lower local error bound for GCP(f,g) if there exists some constant $\bar{\tau}_1 > 0$ such that $\bar{\tau}_1 e(x) \leq dist(x,X^*)$, for each $x \in B$, where $B \subset \Re^n$;

(c) a residual e is upper global error bound for GCP(f,g) if there exists some constant $\tau_2 > 0$ such that $dist(x,X^*) \leq \tau_2 e(x)$, for each $x \in \Re^n$; a residual e is upper local error bound for GCP(f,g) if there exists some constant $\bar{\tau}_2 > 0$ such that $dist(x,X^*) \leq \bar{\tau}_2 e(x)$, for each $x \in B$, where $B \subset \Re^n$.

For each $i \in \{1,2,\ldots,n\}$, define

$$r_i(x) := \min\{f_i(x),g_i(x)\}.$$

Let $r(x)$ denote the vector with components $r_i(x)$, $i \in \{1,2,\ldots,n\}$. $\|r(x)\|$ is called the natural residual for GCP.

First, we show that the Lipschitz continuity is all what we need for $r(x)$ to be a lower error bound for $dist(x,X)$ for any GCP.

Lemma 1[16, *Lemma 7.2]* *Assume that f and g are Lipschitz continuous with constant $L > 0$. Then for all $i \in \{1,2,\dots,n\}$, we have*

$$|r_i(x)| \le L\|x - x^*\|, \quad \text{for all } x \in \Re^n$$

where x^* is an arbitrary solution of GCP(f,g).

Lemma 2[16, *Lemma 7.4]* *Assume that f and g are Lipschitz continuous with constant $L > 0$. Then for each $i \in \{1,2,\dots,n\}$ and any solution $x^* \in \Re^n$ of GCP(f,g), we have*

$$(f_i(x) - f_i(x^*))(g_i(x) - g_i(x^*)) \le 2L|r_i(x)|\|x - x^*\|,$$

for all $x \in \Re^n$.

The proof of the following lemma is similar to Lemma 3.2 and Proposition 3.1 in [3] so we omit the proof.

Lemma 3*Let $\phi_p : \Re^2 \to \Re$ be defined as (6). Then for any $p > 1$, we have*

(i)

$$(2-2^{\frac{1}{p}})|\min\{a,b\}| \le |\phi_p(a,b)| \le (2+2^{\frac{1}{p}})|\min\{a,b\}|.$$

(ii)

$$[2-2^{\frac{1}{p}}]^2\,\Psi_{NR}(x) \le \Psi_p(x) \le [2+2^{\frac{1}{p}}]^2\,\Psi_{NR}(x)$$

for all $x \in R^n$.

Lemma 4*Let $\phi_1 : \Re^2 \to \Re$ be defined as (7). Then for any $p > 1$ and $\alpha > 0$, we have*

(i)

$$\begin{aligned}&(2-2^{\frac{1}{p}})|\min\{a,b\}| \le |\phi_1(a,b)| \\ &\le (2+2^{\frac{1}{p}}+\alpha C_1)|\min\{a,b\}|\end{aligned}$$

for any $a,b \in B_1$, where B_1 is a set such that $\max_{x\in B_1} x \le C_1$.

(ii)

$$[2-2^{\frac{1}{p}}]^2\Psi_{NR}(x) \le \Psi_1(x) \le [2+2^{\frac{1}{p}}+\alpha C_1]^2\,\Psi_{NR}(x)$$

for all $x \in R^n$.

Proof. It is enough to show part (i). When $a > 0, b > 0$, then $\phi_p(a,b) > 0$ and $a_+b_+ > 0$, then we have

$$\begin{aligned}&(2-2^{\frac{1}{p}})|\min\{a,b\}| \le |\phi_1(a,b)| = \phi_1(a,b) \\ &= \phi_p(a,b) + \alpha a_+b_+ \le (2+2^{\frac{1}{p}}+\alpha C_1)|\min\{a,b\}|;\end{aligned}$$

otherwise, $(2-2^{\frac{1}{p}})|\min\{a,b\}| \le |\phi_1(a,b)| = |\phi_p(a,b)| \le (2+2^{\frac{1}{p}})|\min\{a,b\}|$.

Lemma 5*Let $\phi_2 : \Re^2 \to \Re$ be defined as (8). Then for any $p > 1$ and $\alpha > 0$, we have*

(i)

$$|\phi_2(a,b)| \le (2+2^{\frac{1}{p}}+\alpha C_1)|\min\{a,b\}|$$

for any $a,b \in B_1$, where B_1 is a set such that $\max_{x\in B_1}|x| \le C_1$.

(ii)

$$\Psi_2(x) \le [2+2^{\frac{1}{p}}+\alpha C_1]^2\Psi_{NR}(x)$$

for all $x \in R^n$.

Proof. Let us show part (i). If $a > 0, b > 0$, then $\phi_p(a,b) > 0$ and $(ab)_+ > 0$, then we have

$$\begin{aligned}&|\phi_2(a,b)| = \phi_1(a,b) = \phi_p(a,b) + \alpha(ab)_+ \\ &\le (2+2^{\frac{1}{p}}+\alpha C_1)|\min\{a,b\}|;\end{aligned}$$

If $a < 0, b < 0$, then $\phi_p(a,b) < 0$ and $(ab)_+ > 0$, then

$$\begin{aligned}&|\phi_2(a,b)| \le |\phi_p(a,b)| + \alpha(ab)_+ \\ &\le (2+2^{\frac{1}{p}}+\alpha C_1)|\min\{a,b\}|\end{aligned}$$

in the other case,

$$|\phi_2(a,b)| = |\phi_p(a,b)| \le (2+2^{\frac{1}{p}})|\min\{a,b\}|.$$

Above all, we have that the conclusion of this Lemma holds.

Remark 1*For ϕ_2, there does not exist a constant $C > 0$ such that $|\phi_2(a,b)| \ge C|\min\{a,b\}|$. Now we give an example. let $a = -1$, $b = -1$, and $\alpha = 2+\sqrt{2}$, then we have $|\phi_2(a,b)| = 0$.*

Lemma 6*Let $\phi_3 : \Re^2 \to \Re$ be defined as (9). Then for any $p > 1$ and $\alpha > 0$, we have*

(i)

$$\begin{aligned}&(2-2^{\frac{1}{p}})|\min\{a,b\}| \le \phi_3(a,b) \\ &\le \sqrt{(2+2^{\frac{1}{p}})^2+\alpha C_1^2}|\min\{a,b\}|\end{aligned}$$

for any $a,b \in B_1$, where B_1 is a set such that $\max_{x\in B_1}|x| \le C_1$.

(ii)

$$\begin{aligned}&[2-2^{\frac{1}{p}}]^2\,\Psi_{NR}(x) \le \Psi_3(x) \\ &\le [(2+2^{\frac{1}{p}})^2+\alpha C_1^2]\,\Psi_{NR}(x) \ \text{ for all } x \in R^n.\end{aligned}$$

Proof. We will prove part (i). When $a > 0, b > 0$, then $\phi_p(a,b) > 0$ and $(ab)_+ > 0$, then we have

$$\begin{aligned}&(2-2^{\frac{1}{p}})|\min\{a,b\}| \le \phi_p(a,b) \le \phi_3(a,b) \\ &= \sqrt{\phi_p(a,b)^2+\alpha(a_+b_+)^2} \\ &\le \sqrt{\phi_p(a,b)^2+\alpha\max\{|a|,|b|\}^2|\min\{a,b\}|^2} \\ &\le \sqrt{(2+2^{\frac{1}{p}})^2+\alpha C_1^2}|\min\{a,b\}|;\end{aligned}$$

otherwise,

$$\begin{aligned}&(2-2^{\frac{1}{p}})|\min\{a,b\}| \le |\phi_3(a,b)| = |\phi_p(a,b)| \\ &\le (2+2^{\frac{1}{p}})|\min\{a,b\}|.\end{aligned}$$

Lemma 7 *Let $\phi_4 : \Re^2 \to \Re$ be defined as (10). Then for any $p > 1$ and $\alpha > 0$, we have*

(i)

$$(2-2^{\frac{1}{p}})|\min\{a,b\}| \le |\phi_4(a,b)| \le \sqrt{(2+2^{\frac{1}{p}})^2+\alpha C_1^2}|\min\{a,b\}|$$

for any $a,b \in B_1$, where B_1 is a set such that $\max_{x\in B_1}|x| \le C_1$.

(ii)

$$[2-2^{\frac{1}{p}}]^2\,\Psi_{NR}(x) \le \Psi_4(x) \le [(2+2^{\frac{1}{p}})^2+\alpha C_1^2]\,\Psi_{NR}(x)$$

for all $x \in R^n$.

Proof. We will show part (i). When $a > 0, b > 0$ or $a < 0, b < 0$, then $(ab)_+ > 0$, then we have

$$\begin{aligned}(2-2^{\frac{1}{p}})|\min\{a,b\}| &\le |\phi_p(a,b)|\\ &\le \phi_4(a,b)\\ &= \sqrt{\phi_p(a,b)^2+\alpha(ab)_+^2}\\ &\le \sqrt{\phi_p(a,b)^2+\alpha\max\{|a|,|b|\}^2|\min\{a,b\}|^2}\\ &\le \sqrt{(2+2^{\frac{1}{p}})^2+\alpha C_1^2}|\min\{a,b\}|;\end{aligned}$$

otherwise,

$$(2-2^{\frac{1}{p}})|\min\{a,b\}| \le |\phi_4(a,b)| = |\phi_p(a,b)| \le (2+2^{\frac{1}{p}})|\min\{a,b\}|.$$

In the following theorems, the merit functions of the GCP functions based on generalized Fisher-Burmeister function and its generalizations provide global error bounds for GCP(f,g) under appropriate conditions.

Recall that when the solution set is nonempty and f and g are relatively uniform **(P)**-functions, the solution of GCP(F, G) is unique [16].

Theorem 1 *Assume that f and g are Lipschitz continuous with constant $L > 0$. Suppose that the solution set X^* of GCP is nonempty. Then for any $p > 1$, we have*

$$\Psi_p(x) \le \frac{(2+2^{\frac{1}{p}})^2}{2} nL^2 dist(x,X^*)^2,$$

for all $x \in \Re^n$.

Proof. By Lemma 1, we have

$$|r_i(x)| \le L\mathrm{dist}(x,X^*), \quad \text{for all} \quad i \in \{1,2,\ldots,n\}.$$

Combining the inequality with Lemma 3, it follows that

$$\begin{aligned}\Psi_p(x) &= \tfrac{1}{2}\textstyle\sum_{i=1}^n|\phi_p(f_i(x),g_i(x))|^2\\ &\le \tfrac{1}{2}\textstyle\sum_{i=1}^n(2+2^{\frac{1}{p}})^2r_i^2(x)\\ &\le \tfrac{(2+2^{\frac{1}{p}})^2}{2}nL^2\mathrm{dist}(x,X^*)^2.\end{aligned}$$

The proof is complete.

Theorem 2 *Assume that f and g are Lipschitz continuous with constant $L > 0$, and that f and g are relatively uniform **(P)**-functions with constant $\eta > 0$. Suppose that the solution set X^* of GCP is nonempty. Then for any $p > 1$, we have*

$$\Psi_p(x) \ge \frac{(2-2^{\frac{1}{p}})^2\eta^2}{8L^2}\|x-x^*\|^2,$$

for all $x \in \Re^n$ and all $x^* \in X^*$.

Proof. From Lemma 2 and the property of f and g being relatively uniform **(P)**-functions, there exists at least one index i_0 such that

$$|r_{i_0}(x)| \ge \frac{\eta}{2L}\|x-x^*\|,$$

which together with Lemma 3 implies that

$$\begin{aligned}\Psi_p(x) &= \tfrac{1}{2}\textstyle\sum_{i=1}^n\phi_p^2(f_i(x),g_i(x))\\ &\ge \tfrac{1}{2}\phi_p^2(f_{i_0}(x),g_{i_0}(x))\\ &\ge \tfrac{(2-2^{\frac{1}{p}})^2}{2}|r_{i_0}(x)|^2\\ &\ge \tfrac{(2-2^{\frac{1}{p}})^2\eta^2}{8L^2}\|x-x^*\|^2.\end{aligned}$$

We get the results.

Theorem 3 *Assume that f and g are Lipschitz continuous with constant $L > 0$. Suppose that the solution set X^* of GCP is nonempty. Then for any $p > 1$ and $\alpha > 0$, there exists a constant $C > 0$ such that*

$$\Psi_1(x) \le C dist(x,X^*)^2, \quad \text{for all } x \in B,$$

where B is a bounded set.

Proof. Since f and g are continuous and B is bounded, then $\exists C_1 > 0$ such that

$$|\max_{1\le i\le n}\{|f_i(x)|,|g_i(x)|\}| \le C_1, \quad \text{for all } x \in B. \tag{13}$$

From the definition of ϕ_1 in (7) and Lemma 3, for any $a,b \in \Re$ and $\max\{|a|,|b|\} \le C_1$, we have

$$\begin{aligned}&|\phi_1(a,b)|\\ &\le |\phi_p(a,b)|+\alpha a_+b_+\\ &\le (2+2^{\frac{1}{p}}+\alpha C_1)|\min\{a,b\}|\end{aligned} \tag{14}$$

where the first inequality and the last inequality are from

$$0 \le a_+b_+ \le |ab| \le \max\{|a|,|b|\}|\min\{a,b\}| \le C_1|\min\{a,b\}|.$$

Combining the inequality in Lemma 1, it is followed that

$$\begin{aligned}\Psi_1(x) &= \tfrac{1}{2}\textstyle\sum_{i=1}^n|\phi_1(f_i(x),g_i(x))|^2\\ &\le \tfrac{1}{2}(2+2^{\frac{1}{p}}+\alpha C_1)^2\textstyle\sum_{i=1}^n r_i^2(x)\\ &\le \tfrac{1}{2}(2+2^{\frac{1}{p}}+\alpha C_1)^2nL^2\mathrm{dist}(x,X^*)^2.\end{aligned}$$

for any $x \in B$. Denote $C := \frac{1}{2}(2+2^{\frac{1}{p}}+\alpha C_1)^2nL^2$, then $C > 0$ and $\Psi_1(x) \le C\mathrm{dist}(x,X^*)^2$ for any $x \in B$.

Theorem 4 *Assume that f and g are Lipschitz continuous with constant $L > 0$. Suppose that the solution set X^* of GCP is nonempty. Then for any $p > 1$ and $\alpha > 0$, there exist constants $C_i > 0, i = 2,3,4$ such that*

$$\Psi_i(x) \le C_i dist(x, X^*)^2, \; i = 2,3,4, \text{ for all } x \in B,$$

where B is a bounded set.

Proof. By a similar proof as in Theorem 3, we can get the results.

Theorem 5 *Assume that f and g are Lipschitz continuous with constant $L > 0$, and that f and g are relatively uniform* **(P)***-functions with constant $\eta > 0$. Suppose that the solution set X^* of GCP is nonempty. Then for any $p > 1, \alpha > 0$, there exist constants $\bar{C}_i > 0, i = 1,3,4$ such that*

$$\Psi_i(x) \ge \bar{C}_i \|x - x^*\|^2, \quad \text{for all } x \in B \text{ and all } x^* \in X^*.$$

where B is a bounded set.

Proof. By a similar proof as Theorem 2, we can get

$$\bar{C}_1 = \bar{C}_3 = \bar{C}_4 = \frac{(2 - 2^{\frac{1}{p}})^2 \eta^2}{8L^2}.$$

The proof of part (i) in the following lemma will be similar to Lemma 3.1 [10] (for NCP context) so we omit the proof.

Lemma 8 *Let $\phi_{\theta,p} : \Re^2 \to \Re$ be defined in (11). Then, for any $p > 1$ and $\theta \in (0,1]$,*

(i)

$$\left(2 - (2\theta)^{\frac{1}{p}}\right) |\min\{a,b\}| \le |\phi_{\theta,p}(a,b)| \le \left(2 + (2\theta)^{\frac{1}{p}}\right) |\min\{a,b\}|.$$

(ii)

$$\left[2 - (2\theta)^{\frac{1}{p}}\right]^2 \Psi_{NR}(x) \le \Psi_{\theta,p}(x) \le \left[2 + (2\theta)^{\frac{1}{p}}\right]^2 \Psi_{NR}(x).$$

Lemma 9 *Let $\psi_{\alpha,\theta,p} : \Re^2 \to \Re$ be defined as (12). Then for any $p > 1$, $\theta \in (0,1]$ and $\alpha \ge 0$, we have*

$$\frac{(2-(2\theta)^{\frac{1}{p}})^2}{2} |\min\{a,b\}|^2 \le \psi_{\alpha,\theta,p} \le \frac{(2+(2\theta)^{\frac{1}{p}})^2 + \alpha C_1^2}{2} |\min\{a,b\}|^2$$

i.e.,

$$\frac{(2-(2\theta)^{\frac{1}{p}})^2}{2} \Psi_{NR}(x) \le \Psi_{\alpha,\theta,p}(x) \le \frac{(2+(2\theta)^{\frac{1}{p}})^2 + \alpha C_1^2}{2} \Psi_{NR}(x).$$

for any $a,b \in B_1$, where B_1 is a set such that $\max_{x \in B_1} |x| \le C_1$.

Proof. Since for any $a,b \in \Re$ and $\max\{|a|,|b|\} \le C_1$, where C_1 is defined in (13),

$$[(ab)_+]^2 \le a^2 b^2 \le (\max\{|a|,|b|\})^2 (\min\{a,b\})^2 \le C_1^2 (\min\{a,b\})^2 \tag{15}$$

Combining (15) and Lemma 8, and with the definition in (12), we have

$$\psi_{\alpha,\theta,p}(a,b) = \frac{\alpha}{2}[(ab)_+]^2 + \frac{1}{2}\phi_{\theta,p}(a,b)^2 \le \frac{1}{2}\left(\alpha C_1^2 + \left(2 + (2\theta)^{\frac{1}{p}}\right)^2\right)(\min\{a,b\})^2$$

for any $a,b \in \Re$ and $\max\{|a|,|b|\} \le C_1$.

Theorem 6 *Assume that f and g are Lipschitz continuous with constant $L > 0$. Suppose that the solution set X^* of GCP is nonempty. Then for any $p > 1$ and $\theta \in (0,1]$, we have*

$$\Psi_{\theta,p}(x) \le \frac{(2+(2\theta)^{\frac{1}{p}})^2}{2} n L^2 dist(x, X^*)^2,$$

for all $x \in \Re^n$.

Proof. By Lemma 8, we have

$$|r_i(x)| \le L\mathrm{dist}(x, X^*), \quad \text{for all} \quad i \in \{1,2,\ldots,n\}.$$

Combining the inequality with Lemma 3, it follows that

$$\begin{aligned}\Psi_{\theta,p}(x) &= \tfrac{1}{2}\textstyle\sum_{i=1}^n |\phi_{\theta,p}(f_i(x), g_i(x))|^2 \\ &\le \tfrac{1}{2}\textstyle\sum_{i=1}^n (2 + (2\theta)^{\frac{1}{p}})^2 r_i^2(x) \\ &\le \tfrac{(2+(2\theta)^{\frac{1}{p}})^2}{2} n L^2 \mathrm{dist}(x, X^*)^2.\end{aligned}$$

The proof is complete.

Theorem 7 *Assume that f and g are Lipschitz continuous with constant $L > 0$, and that f and g are relatively uniform* **(P)***-functions with constant $\eta > 0$. Suppose that the solution set X^* of GCP is nonempty. Then for any $p > 1$, we have*

$$\Psi_{\theta,p}(x) \ge \frac{(2-(2\theta)^{\frac{1}{p}})^2 \eta^2}{8L^2} \|x - x^*\|^2,$$

for all $x \in \Re^n$ and all $x^* \in X^*$.

Proof. From Lemma 2 and the property of f and g being relatively uniform **(P)**-functions, there exists at least one index i_0 such that

$$|r_{i_0}(x)| \ge \frac{\eta}{2L} \|x - x^*\|,$$

which together with Lemma 3 implies that

$$\begin{aligned}\Psi_{\theta,p}(x) &= \tfrac{1}{2}\textstyle\sum_{i=1}^n \phi_p^2(f_i(x), g_i(x)) \\ &\ge \tfrac{1}{2}\phi_p^2(f_{i_0}(x), g_{i_0}(x)) \\ &\ge \tfrac{(2-(2\theta)^{\frac{1}{p}})^2}{2} |r_{i_0}(x)|^2 \\ &\ge \tfrac{(2-(2\theta)^{\frac{1}{p}})^2 \eta^2}{8L^2} \|x - x^*\|^2.\end{aligned}$$

Theorem 8*Assume that f and g are Lipschitz continuous with constant $L > 0$. Suppose that the solution set X^* of GCP is nonempty. Then any $p > 1$, $\theta \in (0,1]$ and $\alpha \geq 0$, there exists a constant $\mathscr{C} > 0$ such that*

$$\Psi_{\alpha,\theta,p}(x) \leq \mathscr{C} dist(x, X^*)^2, \quad \text{for all } x \in B,$$

where B is a bounded set.

Proof. From Lemma then for all $x \in B$,

$$\begin{aligned}\Psi_{\alpha,\theta,p}(x) &= \sum_{i=1}^n \psi_{\alpha,\theta,p}(f_i(x), g_i(x)) \\ &\leq \frac{1}{2}\sum_{i=1}^n \left(\alpha C_1^2 + \left(2 + (2\theta)^{\frac{1}{p}}\right)^2\right) r_i^2(x) \\ &\leq \frac{\alpha C_1^2 + \left(2+(2\theta)^{\frac{1}{p}}\right)^2}{2} nL^2 \text{dist}(x, X^*)^2.\end{aligned}$$

for any $x \in B$. Denote $\mathscr{C} := \frac{\alpha C_1^2 + \left(2+(2\theta)^{\frac{1}{p}}\right)^2}{2} nL^2$, then $\mathscr{C} > 0$ and $\Psi_{\alpha,\theta,p}(x) \leq L\text{dist}(x, X^*)^2$ for any $x \in B$.

Theorem 9*Assume that f and g are Lipschitz continuous with constant $L > 0$, and that f and g are relatively uniform* **(P)***-functions with constant $\eta > 0$. Suppose that the solution set X^* of GCP is nonempty. Then for any $p > 1$, $\theta \in (0,1]$ and $\alpha \geq 0$, we have*

$$\Psi_{\alpha,\theta,p}(x) \geq \frac{(2-(2\theta)^{\frac{1}{p}})^2 \eta^2}{8L^2} dist(x, X^*)^2, \quad \text{for all } x \in B,$$

where B is a bounded set.

Proof. The proof can be obtained in a similar way.

Corollary 1*Under the assumptions that f and g are Lipschitz continuous and f and g are relatively uniform* **(P)***-functions, the functions $\sqrt{\Psi_p(x)}$ and $\sqrt{\Psi_{\theta,p}(x)}$ provide lower and upper global error bounds for GCP(f,g); the functions $\sqrt{\Psi_1(x)}$, $\sqrt{\Psi_3(x)}$, $\sqrt{\Psi_4(x)}$ and $\sqrt{\Psi_{\alpha,\theta,p}(x)}$ provide lower and upper local error bounds for GCP(f,g).*

Final Remarks

In this paper, we established global error bounds on the generalized complementarity problem, GCP(f,g), based on generalized Fischer-Burmeister function and its generalizations which are not only the extensions of those for the classical nonlinear complementarity problems but also new results for the nonlinear/generalized complementarity problems. For example, our results give various results for generalized complementarity problem when p-norm replace by 2-norm (or when p is an integer greater than 2). Also, when $g(x) = x$, our results further give a unified/generalization treatment of such results for the nonlinear complementarity problem based on generalized Fisher-Burmeister function and its generalizations.

Definitely, we may use the error bound estimation to establish quick convergence rate of the Newton-type method and derivative free algorithms for solving the GCP(f,g), this is a topic for future research by the authors. The results obtained in this paper can be taken as an extension of the existing global error bound for the classical nonlinear complementarity problems.

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