Appl. Math. Inf. Sci. **10**, No. 1, 33-48 (2016)

Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/100104

Exponential Parameterization and ε-Uniformly Sampled Reduced Data

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Received: 7 Jun. 2015, Revised: 5 Aug. 2015, Accepted: 6 Aug. 2015
Published online: 1 Jan. 2016

Abstract: We study the quality of piecewise-quadratic Lagrange interpolation for nonparametric data based on ε-uniform sampling and different forms of exponential parameterization. Surprisingly, it turns out that there is a sharp discontinuity in the quality of interpolation: exponential parameterization performs no better than a blind uniform guess, except for the case of scaled cumulative chord, which matches parametric interpolation.

Keywords: Curve interpolation, numerical analysis, asymptotics, exponential parameterization, different samplings

1 Introduction

A list of $m+1$ points $Q_m = (q_0, q_1, \ldots, q_m)$ in Euclidean n-space E^n is obtained by sampling an unknown but sufficiently smooth and regular curve $\gamma : [0,1] \to E^n$ at $0 = t_0 < t_1 < t_2 < \ldots < t_m = 1$, where $t_1, t_2, \ldots, t_{m-1}$ are also unknown. Here $q_i = \gamma(t_i)$ for $0 \leq i \leq m$, and we have a problem of *nonparametric interpolation* (see e.g. [1]). More precisely, the task is to estimate the unknown curve γ by a curve $\hat{\gamma} : [0,1] \to E^m$ such that $\hat{\gamma}(\hat{t}_i) = q_i$ for all $i = 0, 1, \ldots, m$, where $\hat{\gamma}$ and the $\hat{t}_i$ are computed from $q_0, q_1, \ldots, q_m$. To emphasize that the $\{t_i\}_{i=0}^m$ are not given, we call $\{q_i\}_{i=0}^m$ the *nonparametric data*. Applications of nonparametric data interpolation in computer vision, computer graphics, engineering or physics can be found in e.g. [2], [3], [4] or [5].

By contrast, when both $\{t_i\}_{i=0}^m$ and $\{q_i\}_{i=0}^m$ are known, the curve γ can be estimated using standard methods for *parametric interpolation*, such as piecewise r-degree Lagrange interpolation. So our task can be performed by a parametric interpolant using estimates $\hat{t}_i$ of the t_i. For this to be useful, we also need to prove results about the quality of the corresponding estimate $\hat{\gamma}$ of the unknown curve γ. Such results will depend on the $\{t_i\}_{i=0}^m$. For instance in the trivial case, when the $\{t_i\}_{i=0}^m$ are chosen uniformly along $[0,1]$ (or otherwise actually known), then $\hat{\gamma}$ is just a parametric interpolant whose properties are known from classical results. Indeed, for $\{t_i\}_{i=0}^m$ satisfying the *admissibility condition*:

$$\lim_{m\to\infty} \delta_m = 0, \quad \text{where} \quad \delta_m = \max_{0\leq i\leq m-1} (t_{i+1} - t_i), \tag{1}$$

there is the well-known result [6]:

Theorem 1. Let $\gamma : [0,1] \to E^n$ be C^{r+1}, where $r \geq 0$ and be *regular* in the sense that $\dot{\gamma}$ is nowhere $\mathbf{0}$. Then piecewise r-degree Lagrange interpolation yields a sharp estimate:

$$\hat{\gamma}(t) = \gamma(t) + O(\delta_m^{r+1}) \tag{2}$$

uniformly in $t \in [0,1]$. $\square$

The asymptotic estimate in (2) is sharp, i.e. there exist $\gamma \in C^{r+1}$ and admissible sampling $\{t_i\}_{i=0}^m$, for which the convergence order established in (2) cannot be improved.

Remark 1. Recall that, for a family $F_{\delta_m} : [0,T] \to E^n$ with $0 < T < \infty$ (e.g. for $F_{\delta_m} = \tilde{\gamma}_r - \gamma$ and $T = 1$; here $\tilde{\gamma}_r$ depends on δ_m) we write $F_{\delta_m} = O(\delta_m^\alpha)$ when $\|F_{\delta_m}\|_\infty = O(\delta_m^\alpha)$, where $\|F_{\delta_m}\|_\infty = \sup_{t\in[0,T]} \|F_{\delta_m}(t)\|$ and $\|\cdot\|$ denotes the Euclidean norm. The latter holds if there exists constant $K > 0$ such that for some $\bar{\delta} > 0$ we have $\|F_{\delta_m}\| \leq K\delta_m^\alpha$, for all $\delta_m \in (0, \bar{\delta})$ and all $t \in [0,T]$. Here K depends on γ and on each sampling $\{t_i\}_{i=0}^m$. Evidently as interval $[0,T]$ is compact once F_{δ_m} is continuous we have $\|F_{\delta_m}\|_\infty = \max_{t\in[0,T]} \|F_{\delta_m}(t)\|$. $\square$

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In our situation, where less information is available about the distribution of the $\{t_i\}_{i=0}^m$, it is natural that $\hat{\gamma}$ should be a lower-quality estimate of γ.

Definition 1. *We say that the* $\{t_i\}_{i=0}^m$ *is sampled* more-or-less uniformly *(see e.g. [3], [7] or [8]) when, for some* $\beta \in (0,1]$*, and all sufficiently large* m *and all* $i = 1,2,\ldots,m$*, we have:*

$$\beta \delta_m \leq t_i - t_{i-1} \leq \delta_m. \tag{3}$$

Equivalently

$$\frac{\beta_0}{m} \leq t_i - t_{i-1} \leq \frac{\beta_1}{m}, \tag{4}$$

for some $0 < \beta_0 \leq \beta_1$*, sufficiently large* m *and all* $i = 1,2,\ldots,m$*. Necessarily* $\beta_1 \geq 1$*, by summing the inequalities.* □

Definition 2. *Given* $\varepsilon > 0$*, we say that* $\{t_i\}_{i=0}^m$ *is* sampled ε-uniformly *(see e.g. [9]) when, for some* C^∞ *diffeomorphism* $\phi : [0,1] \to [0,1]$*, sufficiently large* m *and all* $0 \leq i \leq m$*,*

$$t_i = \phi\left(\frac{i}{m}\right) + O\left(\frac{1}{m^{1+\varepsilon}}\right). \tag{5}$$

This is more restrictive than the condition enforcing $\{t_i\}_{i=0}^m$ *to be distributed more-or-less uniformly. Since by (1),* $m\delta_m \geq 1$ *and thus the second term in (5) reads as* $O(\delta_m^{1+\varepsilon})$*.* □

Again both ϕ and the $O(\delta_m^{1+\varepsilon})$ term depend on the ε-uniform sampling. The most common method to estimate the unknown knots $\{t_i\}_{i=0}^m$ from the nonparametric data is to use some form of *exponential parametrization* (see e.g. [4]) in the following sense:

Definition 3. *Choose* $\lambda \in [0,1]$ *and set* $\hat{t}_0 = 0$*. Then, inductively, for* $1 \leq i \leq m$*, set*

$$\hat{t}_i = \hat{t}_{i-1} + \|q_i - q_{i-1}\|^\lambda. \tag{6}$$

Finally, set normalized $\tilde{t}_i = \hat{t}_i/\hat{t}_m$*, for* $0 \leq t \leq m$*. In order to ensure* $\tilde{t}_i < \tilde{t}_{i+1}$ *(and also that* $\hat{t}_i < \hat{t}_{i+1}$*) we assume that* $q_i \neq q_{i+1}$*.* □

The choice $\lambda = 0$ yields $\hat{t}_i = i$, corresponding to a blind uniform guess, taking no account of the spread of interpolation points $\{q_i\}_{i=0}^m$ (see [9]).

Theorem 2. Let γ be C^3 and let the unknown $\{t_i\}_{i=0}^m$ be sampled ε-uniformly, where $\varepsilon > 0$. If $\hat{\gamma}$ is constructed using piecewise-quadratic Lagrange interpolation based on $\lambda = 0$ (blind uniform guess) then, for piecewise-C^∞ reparameterization $\psi : [0,1] \to [0,1]$ (computed from data Q_m), we have sharp asymptotic estimate over $[0,1]$:

$$(\hat{\gamma} \circ \psi)(t) = \gamma(t) + O(\delta_m^{\min\{3,1+2\varepsilon\}}),$$

for the trajectory approximation. □

Note that in case of reduced data Q_m for F_{δ_m} (see Remark 1) we substitute $F_{\delta_m} = \hat{\gamma}_2 \circ \psi - \gamma$.

At the other extreme we have a more informative estimate of the $\{t_i\}_{i=0}^m$, namely the scaled *cumulative chord parameterization* given by exponential parameterization with $\lambda = 1$. Indeed, we have the following (see [10]):

Theorem 3. Let γ be C^3 and let the unknown t_i be sampled ε-uniformly, where $\varepsilon > 0$. If $\hat{\gamma}$ is constructed using piecewise-quadratic Lagrange interpolation based on $\lambda = 1$ (scaled cumulative chord) then, for piecewise-C^∞ reparameterization $\psi : [0,1] \to [0,1]$ (computed from data Q_m), the sharp asymptotic estimate:

$$(\hat{\gamma} \circ \psi)(t) = \gamma(t) + O(\delta_m^3) \tag{7}$$

follows for $t \in [0,1]$. In fact (7) holds also for arbitrary admissible samplings (1). □

So scaled cumulative chord parametrization performs as well as the parametric interpolant obtained by taking $r = 2$ in Th. 1, at least in terms of asymptotic and modulo the reparameterization ψ. On the other hand, the asymptotics for the blind uniform guess of Th. 2 are not nearly so good for small values of ε. Between these extremes, one might expect a steady increase in the exponent of δ_m (or of $1/m$) as λ increases from 0 to 1. Surprisingly this does not happen, as shown in Th. 4 below, which *is the main result of this paper*:

Theorem 4. Let γ be C^4 and let the unknown t_i be sampled ε-uniformly where $\varepsilon > 0$. If $\hat{\gamma}$ is constructed using piecewise-quadratic Lagrange interpolation based on $\lambda \in (0,1)$ then, for some piecewise-quadratic-C^∞ reparameterization $\psi : [0,1] \to [0,1]$ (computed from data Q_m):

$$(\hat{\gamma} \circ \psi)(t) = \gamma(t) + O(\delta_m^{\min\{3,1+2\varepsilon\}}) \tag{8}$$

holds for $t \in [0,1]$. □

A similar phenomenon is discovered in [11] for more-or-less uniform samplings, where $(\hat{\gamma} \circ \psi)(t) = \gamma(t) + O(\delta_m)$, for $\lambda \in [0,1)$ and $(\hat{\gamma} \circ \psi)(t) = \gamma(t) + O(\delta_m^3)$, for either $\lambda = 1$, or a uniform sampling and $\lambda \in [0,1)$.

This paper proves Th. 4 and its sharpness in Ex. 2. The general framework used here, has some similarities to [11] which applies only to such more-or-less uniformly sampled curves for which $\psi : [0,1] \to [0,1]$ is a reparameterization. Our proof for ε-uniform samplings is different and also ensures that $\dot{\psi} > 0$ holds for curves $\gamma \in C^3$. Also, for the more restrictive case of samplings (5) we achieved a better trajectory approximation than for the general class of more-our-less uniform samplings established in [11]. As well as the analysis of the Ex. 2 the numerical tests confirm the sharpness of the asymptotics from Th. 4.

2 Exponential parameterization for ε-uniform samplings

The following example is used later in proving Th. 4.

Example 1. a) An inspection reveals that each ε-uniform sampling is also more-or-less uniform. Indeed by Taylor's Th. and (5) we have:

$$t_{i+1}-t_i = \dot{\phi}(\frac{iT}{m})\frac{T}{m}+O(\frac{1}{m^{1+\varepsilon}})+O(\frac{1}{m^2})$$
$$= \dot{\phi}(\frac{iT}{m})\frac{T}{m}+O(\frac{1}{m^{\min\{2,1+\varepsilon\}}}). \quad (9)$$

Since $\varepsilon>0$ we have $m^{-\min\{2,1+\varepsilon\}}<m^{-1}$ and thus by boundedness of continuous $\dot{\phi}$ over compact $[0,1]$, there exist constants $0<K_l<K_u$ such that:

$$\frac{K_l}{m}\leq t_{i+1}-t_i\leq\frac{K_u}{m}.$$

So ε-uniformity implies more-or-less uniformity (however not conversely).

b) By (5) and Taylor's expansion (applied to ϕ at $t=i/m$) we have for each $\varepsilon>0$ the following (with $j=0,1$):

$$t_{i+j+1}-t_{i+j}=\dot{\phi}\left(\frac{i}{m}\right)\frac{1}{m}+O\left(\frac{1}{m^{\min\{2,1+\varepsilon\}}}\right). \quad (10)$$

Combining (10) with $0<1/m\leq\delta_m$ (as $\sum_{i=0}^{m}(t_{i+1}-t_i)=1$ and thus $m\delta_m\geq1$) gives:

$$t_{i+2}-t_{i+1}=t_{i+1}-t_i+O(\delta_m^{\min\{2,1+\varepsilon\}}). \quad (11)$$

For uniform sampling $\{t_i\}_{i=0}^m$ we have $t_{i+2}-t_{i+1}=t_{i+1}-t_i=\delta=1/m$. $\square$

We pass now to the proof of Th. 4.

Proof. As we see later in Remark 5 it is sufficient to prove the asymptotics (8) for both unnormalized knots $\{\hat{t}_i\}_{i=0}^m$ (see (6)) and shifted according to $\hat{t}-\hat{t}_i$. For simplicity the knots in (6) and (12) use the same notation. Let $\psi_i:I_i=[t_i,t_{i+2}]\to\hat{I}_i=[\hat{t}_i,\hat{t}_{i+2}]$ be the quadratic polynomial satisfying interpolation conditions $\psi_i(t_{i+j})=\hat{t}_{i+j}$, with $j=0,1,2$, where

$$\hat{t}_i=0,\quad \hat{t}_{i+1}=\|q_{i+1}-q_i\|^{\lambda},$$
$$\hat{t}_{i+2}=\hat{t}_{i+1}+\|q_{i+2}-q_{i+1}\|^{\lambda}. \quad (12)$$

The track-sum of $\{\psi_i\}_{i=0}^{m-2}$ (for $i=0,2,4,\ldots,m-2$) defines a continuous piecewise-C^∞ mapping $\psi:[0,1]\to[0,\hat{T}]$, where $\hat{T}=\hat{t}_m$.

The proof of Th. 4 is divided into five steps:

2.1 Step 1: proof that ψ is a reparameterization

We show first that ψ_i is asymptotically a reparameterization of I_i into $\hat{I}_i$, for arbitrary $\varepsilon>0$ and $\lambda\in[0,1]$. This is proved here under the weaker assumption that $\gamma\in C^3([0,1])$ - recall that by [11], for either $\lambda=1$ and $\{t_i\}_{i=0}^m$ merely admissible (1) or $\{t_i\}_{i=0}^m$ uniform and $\lambda\in[0,1)$, the quadratic ψ_i yields also asymptotically a reparameterization. This is not always true for arbitrary more-or-less uniform samplings (3) and $\lambda\in[0,1)$ as both shown also in [11].

Newton's Interpolation Formula for divided differences $\psi_i[\cdot]$, $\psi_i[\cdot,\cdot]$ and $\psi_i[\cdot,\cdot,\cdot]$ (see [6]) gives over each I_i:

$$\psi_i(t) = \psi_i[t_i]+\psi_i[t_i,t_{i+1}](t-t_i)$$
$$+\psi_i[t_i,t_{i+1},t_{i+2}](t-t_i)(t-t_{i+1}),$$
$$\psi_i^{(1)}(t) = \psi_i[t_i,t_{i+1}]+(2t-t_{i+1}-t_i)\psi_i[t_i,t_{i+1},t_{i+2}],$$
$$\psi_i^{(2)}(t) = 2\psi_i[t_i,t_{i+1},t_{i+2}]. \quad (13)$$

For ψ_i to be a reparameterization it suffices to show that $\psi_i^{(1)}>0$ over I_i. For the latter, as $\psi_i^{(1)}(t)$ is linear, it is sufficient to demonstrate that both $\psi_i^{(1)}(t_i)>0$ and $\psi_i^{(1)}(t_{i+2})>0$ hold asymptotically. In doing so, by (13) a simple inspection reveals:

$$\psi_i^{(1)}(t_i) = \psi_i[t_i,t_{i+1}]+(t_i-t_{i+1})\psi_i[t_i,t_{i+1},t_{i+2}],$$
$$\psi_i^{(1)}(t_{i+2}) = \psi_i[t_i,t_{i+1}]$$
$$+((t_{i+2}-t_{i+1})+(t_{i+2}-t_i))\psi_i[t_i,t_{i+1},t_{i+2}]. \quad (14)$$

To show inequality $\psi_i^{(1)}(t_i)>0$, recall (see [11]) that $\gamma\in C^3([0,T])$ with formula (1) leads to:

$$\psi_i[t_i,t_{i+1}] = (t_{i+1}-t_i)^{-1+\lambda}+O((t_{i+1}-t_i)^{1+\lambda})$$
$$= (t_{i+1}-t_i)^{-1+\lambda}+O(\delta_m^{1+\lambda}),$$
$$\psi_i[t_{i+1},t_{i+2}] = (t_{i+2}-t_{i+1})^{-1+\lambda}+O((t_{i+2}-t_{i+1})^{1+\lambda})$$
$$= (t_{i+2}-t_{i+1})^{-1+\lambda}+O(\delta_m^{1+\lambda}),$$
$$\psi_i[t_i,t_{i+1},t_{i+2}] = \frac{(t_{i+2}-t_{i+1})^{-1+\lambda}-(t_{i+1}-t_i)^{-1+\lambda}}{t_{i+2}-t_i}$$
$$+O(\delta_m^{\lambda}). \quad (15)$$

We examine now the asymptotics of the second term of $\psi_i^{(1)}(t_i)$ in (14) (denoted below as J_i) by using the definition of the second divided differences $\psi_i[t_i,t_{i+1},t_{i+2}]$:

$$J_i = -(t_{i+1}-t_i)\psi_i[t_i,t_{i+1},t_{i+2}]$$
$$= -\frac{t_{i+1}-t_i}{t_{i+2}-t_i}\left(\psi_i[t_{i+1},t_{i+2}]-\psi_i[t_i,t_{i+1}]\right). \quad (16)$$

Combining (16) with $0<(t_{i+1}-t_i)(t_{i+2}-t_i)^{-1}<1$ (a term of order $O(1)$ with non vanishing asymptotic constant) and with (15) and finally coupling it with (5) (thus yielding (11)) leads to:

$$J_i = O(1)[(t_{i+2}-t_{i+1})^{-1+\lambda}-(t_{i+1}-t_i)^{-1+\lambda}+O(\delta_m^{1+\lambda})]$$
$$= O(1)[((t_{i+1}-t_i)+O(\delta_m^{\min\{2,1+\varepsilon\}}))^{-1+\lambda}$$
$$-(t_{i+1}-t_i)^{-1+\lambda}+O(\delta_m^{1+\lambda})]$$
$$= O(1)$$
$$\cdot[(t_{i+1}-t_i)^{-1+\lambda}(1+(t_{i+1}-t_i)^{-1}O(\delta_m^{\min\{2,1+\varepsilon\}}))^{-1+\lambda}$$
$$-(t_{i+1}-t_i)^{-1+\lambda}+O(\delta_m^{1+\lambda})].$$

As any ε-uniform sampling is also more-or-less uniform (see Ex. 1) the following holds $(t_{i+1}-t_i)^{-1}=O(\delta_m^{-1})$ and hence:

$$J_i = O(1)[(t_{i+1}-t_i)^{-1+\lambda}(1+O(\delta_m^{\min\{1,\varepsilon\}}))^{-1+\lambda}$$
$$-(t_{i+1}-t_i)^{-1+\lambda}+O(\delta_m^{1+\lambda})]. \tag{17}$$

By Taylor's expansion we obtain that $(1+x)^{-1+\lambda}=1+(-1+\lambda)(1+\xi)^{-(2-\lambda)}x$, where $|\xi|\leq|x|$. Setting $x=O(\delta_m^{\min\{1,\varepsilon\}})$ and taking into account that $2-\lambda>0$, we have $(1+\xi)^{-2+\lambda}=O(1)$ (as ξ is asymptotically separated from -1). Consequently, $(1+O(\delta_m^{\min\{1,\varepsilon\}}))^{-1+\lambda}=1+O(\delta_m^{\min\{1,\varepsilon\}})$, which in turn coupled with (17) gives (with the term $O(1)$ having non-vanishing asymptotic constant):

$$J_i = O(1)[(t_{i+1}-t_i)^{-1+\lambda}(1+O(\delta_m^{\min\{1,\varepsilon\}}))$$
$$-(t_{i+1}-t_i)^{-1+\lambda}+O(\delta_m^{1+\lambda})]$$
$$= O(1)[O(\delta_m^{\min\{\lambda,-1+\lambda+\varepsilon\}})+O(\delta_m^{1+\lambda})]$$
$$= O(\delta_m^{\min\{\lambda,-1+\lambda+\varepsilon,1+\lambda\}})$$
$$= O(\delta_m^{\min\{\lambda,-1+\lambda+\varepsilon\}})$$
$$= \begin{cases} O(\delta_m^{-1+\lambda+\varepsilon}), & \text{for } 0<\varepsilon\leq 1;\\ O(\delta_m^{\lambda}), & \text{for } \varepsilon>1.\end{cases} \tag{18}$$

Combining (18) with (14), (15), (16) and $\lambda\in[0,1)$ results in:

$$\psi_i^{(1)}(t_i)$$
$$= (t_{i+1}-t_i)^{-1+\lambda}+O(\delta_m^{1+\lambda})+O(\delta_m^{\min\{\lambda,-1+\lambda+\varepsilon\}})$$
$$= (t_{i+1}-t_i)^{-1+\lambda}+O(\delta_m^{\min\{\lambda,-1+\lambda+\varepsilon\}})>0 \tag{19}$$

asymptotically (as $-1+\lambda<\min\{\lambda,-1+\lambda+\varepsilon\}$, for $\varepsilon>0$ and $1+\lambda>\min\{\lambda,-1+\lambda+\varepsilon\}$). By (14) as $0<[(t_{i+2}-t_i)+(t_{i+2}-t_{i+1})](t_{i+2}-t_i)^{-1}<2$ the above argument analogously justifies the second inequality $\psi_i^{(1)}(t_{i+2})>0$. Hence, asymptotically the mapping ψ_i is a reparametrization of I_i into $\hat{I}_i$. Thus the discussion of *Step 1* is completed.

2.2 Step 2: difference between interpolant $\hat{\gamma}_2$ and curve γ

In order to accelerate the linear convergence rates for trajectory estimation from [11] established for more-or-less uniform samplings (3), $\lambda\in[0,1)$ and any regular curve $\gamma\in C^3([0,1])$ we assume from now on that $\gamma\in C^4([0,1])$.

Let the interpolant $\hat{\gamma}_2(\hat{t}_i)=q_i$ be defined as a track-sum of quadratics $\hat{\gamma}_{2,i}:[\hat{t}_i,\hat{t}_{i+2}]\to E^n$ satisfying $\hat{\gamma}_{2,i}(\hat{t}_{i+j})=q_{i+j}$, for $j=0,1,2$ and $i=2k$, where $k=0,1,\ldots,m/2$. The difference between the interpolant $\hat{\gamma}=\hat{\gamma}_2$ and the unknown curve γ over each I_i (and thus over $[0,1]$ since mapping ψ_i is a reparameterization - see *Step 1*) reads as:

$$f_i(t)=(\hat{\gamma}_{2,i}\circ\psi_i)(t)-\gamma(t). \tag{20}$$

Thus as $\hat{\gamma}_{2,i}(\hat{t}_{i+j})=(\hat{\gamma}_{2,i}\circ\psi)(t_{i+j})$ (for $j=0,1,2$) we arrive at:

$$f_i(t_{i+j})=\mathbf{0}. \tag{21}$$

Recall now Hadamard's Lemma (see [12]; Part 1, Lemma 2.1):

Lemma 1. Let $f:[a,b]\to E^n$ be of class C^l, where $l\geq 1$ and assume that $f(t_0)=\mathbf{0}$, for some $t_0\in(a,b)$. Then there exists a C^{l-1} function $g:[a,b]\to E^n$ for which we have $f(t)=(t-t_0)g(t)$. In addition $g(t)=O(\frac{df}{dt})$. $\square$

In order to construct the function $h(t)$ it suffices to note that $f(t)=F(1)-F(0)$, where $F(u)=f(tu+(1-u)t_0)$. Thus by the Fundamental Th. of Calculus we obtain the following:
$f(t)=\int_0^1 F_u'(u)du=(t-t_0)\int_0^1 f'(tu+(1-u)t_0)du$.

An inspection of the proof of Lemma 1 leads to its generalization with f having multiple zeros $t_0<t_1<\cdots<t_k$. Indeed upon $k+1$ applications of Lemma 1 we obtain:

$$f(t)=(t-t_0)(t-t_1)\ldots(t-t_k)h(t), \tag{22}$$

where h is of class $C^{l-(k+1)}$ and $h=O(\frac{d^{k+1}f}{dt^{k+1}})$.

Consequently, by Hadamard's Lemma, for each $t\in I_i$ we have:

$$f_i(t)=(t-t_i)(t-t_{i+1})(t-t_{i+2})g_i(t), \tag{23}$$

where $g_i(t)=O(f_i^{(3)}(t))$, uniformly over I_i. Furthermore

$$f_i(t)=O(\delta_m^3)\cdot O\left((\hat{\gamma}_{2,i}\circ\psi_i)^{(3)}(t)-\gamma^{(3)}(t)\right). \tag{24}$$

Using the chain rule for the composition of two quadratics $\hat{\gamma}_{2,i} \circ \psi_i$ combined with $\gamma \in C^4([0,1])$, (24) gives[1]:

$$f_i(t) = O(\delta_m^3) \cdot \left(O(\hat{\gamma}''_{2,i}(\hat{t})) \cdot O(\psi_i^{(1)}(t)) \cdot O(\psi_i^{(2)}(t)) + O(1) \right), \quad (25)$$

for $t \in I_i$ and $\hat{t} \in \hat{I}_i$, where $\hat{\gamma}''_{2,i}$ denotes the second derivative of $\hat{\gamma}_{2,i}$ with respect to $\hat{t} = \psi_i(t) \in \hat{I}_i$. In order to examine the asymptotics of (25) it suffices to analyze now the asymptotics of three involved terms, namely $O(\hat{\gamma}''_{2,i}(\hat{t}))$, $O(\psi_i^{(1)}(t))$ and $O(\psi_i^{(2)}(t))$. As to be shown, the respective asymptotic orders of the above three terms are independent from I_i.

2.3 Step 3: asymptotic orders of $\psi^{(k)}(t)$, $k = 1, 2$

First we discuss the asymptotics of $O(\psi_i^{(1)}(t))$ and $O(\psi_i^{(2)}(t))$, given $\lambda \in [0,1)$ and (5). In doing so it suffices to analyze asymptotic orders of two divided differences $\psi_i[t_i, t_{i+1}]$ and $\psi_i[t_i, t_{i+1}, t_{i+2}]$, respectively.

By Taylor's Th. and $\gamma \in C^4([0,T])$, for each $t \in I_i$ we have:

$$\gamma(t) = \sum_{k=0}^{3} \frac{\gamma^{(k)}(t_i)}{k!}(t-t_i)^k + O\left((t-t_i)^4\right). \quad (26)$$

Furthermore by (12) the following holds:

$$\psi_i[t_i, t_{i+1}] = \frac{\psi_i(t_{i+1}) - \psi_i(t_i)}{t_{i+1} - t_i} = \frac{\left(\|\gamma(t_{i+1}) - \gamma(t_i)\|^2\right)^{\lambda/2}}{t_{i+1} - t_i}. \quad (27)$$

Since γ is regular (i.e. $\dot{\gamma} \neq \mathbf{0}$), it can be reparameterized to the arc-length parameterization with $\|\gamma^{(1)}(t)\| \equiv 1$ over $[0,1]$ (see e.g. [14]). Such reparameterization does not influence the asymptotics in question. Therefore as $h(t) = \langle \gamma^{(1)}(t) | \gamma^{(1)}(t) \rangle \equiv 1$ over $t \in [0,1]$, (here $\langle \cdot | \cdot \rangle$ denotes a standard dot product in E^n) upon differentiating a constant function $h(t)$ one arrives to:

$$0 = \langle \gamma^{(1)}(t) | \gamma^{(1)}(t) \rangle^{(1)} = 2 \langle \gamma^{(1)}(t) | \gamma^{(2)}(t) \rangle, \quad (28)$$

which in turn results in $\gamma^{(1)}$ and $\gamma^{(2)}$ being mutually orthogonal. Taking the derivative of (28) yields:

$$\langle \gamma^{(1)}(t) | \gamma^{(3)}(t) \rangle = -\langle \gamma^{(2)}(t) | \gamma^{(2)}(t) \rangle = -\kappa^2(t), \quad (29)$$

where $\kappa(t)$ is the curvature of γ at t. Combining $\|\gamma^{(1)}(t)\| = 1$, (26) (evaluated at $t = t_{i+1}$), (28), (29) we obtain $\|\gamma(t_{i+1}) - \gamma(t_i)\|^2 / (t_{i+1} - t_i)^2$

$$= \| \sum_{k=1}^{3} \frac{\gamma^{(k)}(t_i)}{k!}(t_{i+1}-t_i)^{k-1} + O((t_{i+1}-t_i)^3) \|^2$$

$$= \langle \sum_{k=1}^{3} \frac{\gamma^{(k)}(t_i)}{k!}(t_{i+1}-t_i)^{k-1} + O((t_{i+1}-t_i)^3) | \sum_{k=1}^{3} \frac{\gamma^{(k)}(t_i)}{k!}(t_{i+1}-t_i)^{k-1} + O((t_{i+1}-t_i)^3) \rangle$$

$$= 1 + \frac{(t_{i+1}-t_i)^2}{4}\kappa^2(t_i) - \frac{(t_{i+1}-t_i)^2}{3}\kappa^2(t_i) + O((t_{i+1}-t_i)^3)$$

$$= 1 - \frac{(t_{i+1}-t_i)^2}{12}\kappa^2(t_i) + O((t_{i+1}-t_i)^3). \quad (30)$$

Consequently, coupling (27) with (30) leads to:

$$\psi_i[t_i, t_{i+1}] = (t_{i+1}-t_i)^{-1+\lambda} \cdot \left(1 - \frac{(t_{i+1}-t_i)^2}{12}\kappa^2(t_i) + O\left((t_{i+1}-t_i)^3\right)\right)^{\frac{\lambda}{2}}.$$

By Taylor's expansion:

$$(1+x)^{\frac{\lambda}{2}} = 1 + \frac{\lambda x}{2} + \frac{\lambda(\lambda-2)}{4\sqrt{(1+\xi)^{4-\lambda}}}x^2,$$

for $|\xi| \leq |x|$, which satisfies $1 + \frac{\lambda x}{2} + O(x^2)$ (for $x > -1 + \rho$, where $\rho > 0$). The latter used with $x = -((t_{i+1} - t_i)^2/12)\kappa^2(t_i) + O((t_{i+1} - t_i)^3)$ (here $x > -1 + \rho$ holds asymptotically) results in $\psi_i[t_i, t_{i+1}]$

$$= (t_{i+1}-t_i)^{-1+\lambda} \cdot \left(1 - \frac{\lambda(t_{i+1}-t_i)^2}{24}\kappa^2(t_i) + O\left((t_{i+1}-t_i)^3\right)\right)$$

$$= (t_{i+1}-t_i)^{-1+\lambda} \cdot \left(1 - \frac{\lambda(t_{i+1}-t_i)^2}{24}\kappa^2(t_i)\right) + O\left((t_{i+1}-t_i)^{2+\lambda}\right). \quad (31)$$

Note that, since $2 + \lambda > 0$ (here $\lambda \in [0,1)$) and $0 < t_{i+1} - t_i \leq \delta_m$ the last expression $O((t_{i+1} - t_i)^{2+\lambda})$ from (31) can also be substituted by $O(\delta_m^{2+\lambda})$. Similarly, for $\psi_i[t_{i+1}, t_{i+2}]$

$$= (t_{i+2}-t_{i+1})^{-1+\lambda}\left(1 - \frac{\lambda(t_{i+2}-t_{i+1})^2}{24}\kappa^2(t_{i+1})\right) + O\left((t_{i+2}-t_{i+1})^{2+\lambda}\right).$$

The latter combined with $k^2(t_{i+1}) = k^2(t_i) + O(t_{i+1} - t_i)$ yields $\psi_i[t_{i+1}, t_{i+2}]$

$$= (t_{i+2}-t_{i+1})^{-1+\lambda} \cdot \left[1 - \frac{\lambda(t_{i+2}-t_{i+1})^2}{24}\kappa^2(t_i) + O\left((t_{i+2}-t_{i+1})^2(t_{i+1}-t_i)\right)\right] + O\left((t_{i+2}-t_{i+1})^{2+\lambda}\right)$$

$$= (t_{i+2}-t_{i+1})^{-1+\lambda}\left(1 - \frac{\lambda(t_{i+2}-t_{i+1})^2}{24}\kappa^2(t_i)\right)$$

[1] Derivatives over $\hat{t}$ are denoted by apostrophes, whereas calculated over t use superscript notation.

$$+O\left((t_{i+2}-t_{i+1})^{1+\lambda}(t_{i+1}-t_i)\right)+O\left((t_{i+2}-t_{i+1})^{2+\lambda}\right). \tag{32}$$

Combining (31), (32) and $|(t_{i+j+1}-t_{i+j})/(t_{i+2}-t_i)|<1$ (for $j=0,1$) renders $\psi_i[t_i,t_{i+1},t_{i+2}]$

$$=\frac{\psi_i[t_{i+1},t_{i+2}]-\psi_i[t_i,t_{i+1}]}{t_{i+2}-t_i}$$
$$=\frac{(t_{i+2}-t_{i+1})^{-1+\lambda}\left(1-\frac{\lambda(t_{i+2}-t_{i+1})^2}{24}\kappa^2(t_i)\right)}{t_{i+2}-t_i}$$
$$-\frac{(t_{i+1}-t_i)^{-1+\lambda}\left(1-\frac{\lambda(t_{i+1}-t_i)^2}{24}\kappa^2(t_i)\right)}{t_{i+2}-t_i}$$
$$+O\left((t_{i+2}-t_{i+1})^{1+\lambda}\right)+O\left((t_{i+2}-t_{i+1})^{1+\lambda}\right). \tag{33}$$

Again, since $\lambda+1\geq 0$ the last two terms are of order $O(\delta_m^{1+\lambda})$.

The argument applied so-far in *Step 3* does not exploit (5). We invoke now ε-uniformity (5). Indeed, recall that from Ex. 1, ε-uniformity implies more-or-less uniformity. By (11), (33) and $|(t_{i+1+j}-t_{i+j})(t_{i+2}-t_i))^{-1}|\leq 1$ (for $j=0,1$) we have $\psi_i[t_i,t_{i+1},t_{i+2}]$

$$=\frac{(t_{i+2}-t_{i+1})^{-1+\lambda}\left(1-\frac{\lambda(t_{i+2}-t_{i+1})^2}{24}\kappa^2(t_i)\right)}{t_{i+2}-t_i}$$
$$-\frac{(t_{i+1}-t_i)^{-1+\lambda}\left(1-\frac{\lambda(t_{i+1}-t_i)^2}{24}\kappa^2(t_i)\right)}{t_{i+2}-t_i}+O(\delta_m^{1+\lambda})$$
$$=\frac{(t_{i+2}-t_{i+1})^{-1+\lambda}-(t_{i+1}-t_i)^{-1+\lambda}}{t_{i+2}-t_i}$$
$$-\frac{\lambda\kappa^2(t_i)}{24}\frac{((t_{i+2}-t_{i+1})^{1+\lambda}-(t_{i+1}-t_i)^{1+\lambda})}{t_{i+2}-t_i}+O(\delta_m^{1+\lambda})$$
$$=\frac{\left((t_{i+1}-t_i)+O(\delta_m^{\min\{2,1+\varepsilon\}})\right)^{-1+\lambda}-(t_{i+1}-t_i)^{-1+\lambda}}{t_{i+2}-t_i}$$
$$-\frac{\lambda\kappa^2(t_i)}{24}$$
$$\cdot\frac{\left((t_{i+1}-t_i)+O(\delta_m^{\min\{2,1+\varepsilon\}})\right)^{1+\lambda}-(t_{i+1}-t_i)^{1+\lambda}}{t_{i+2}-t_i}$$
$$+O(\delta_m^{1+\lambda}), \tag{34}$$

which by (3) (as any ε-uniform sampling is also more-or-less uniform and thus $t_{i+1}-t_i=O(\delta_m^{-1})$) and by Taylor's expansion of either $(1+x)^{-1+\lambda}=1+(-1+\lambda)(1+\xi)^{-2+\lambda}x=1+O(x)$ or of $(1+x)^{1+\lambda}=1+(1+\lambda)(1+\xi)^{\lambda}x$ (applied at $x_0=0$ and for $x=O(\delta_m^{\min\{1,\varepsilon\}})$ separated from -1 for $\varepsilon>0$, here $|\xi|=O(x)$) yields $\psi_i[t_i,t_{i+1},t_{i+2}]$

$$=\frac{(t_{i+1}-t_i)^{-1+\lambda}\left[\left(1+O(\delta_m^{\min\{1,\varepsilon\}})\right)^{-1+\lambda}-1\right]}{t_{i+2}-t_i}$$
$$-\frac{\lambda\kappa^2(t_i)}{24}\frac{(t_{i+1}-t_i)^{1+\lambda}\left[\left(1+O(\delta_m^{\min\{1,\varepsilon\}})\right)^{1+\lambda}-1\right]}{t_{i+2}-t_i}$$
$$+O(\delta_m^{1+\lambda})$$
$$=\frac{(t_{i+1}-t_i)^{-1+\lambda}(\lambda-1)O(\delta_m^{\min\{1,\varepsilon\}})}{t_{i+2}-t_i}+O(\delta_m^{1+\lambda})$$
$$-\frac{\lambda(1+\lambda)\kappa^2(t_i)}{24}\frac{(t_{i+1}-t_i)^{1+\lambda}O(\delta_m^{\min\{1,\varepsilon\}})}{t_{i+2}-t_i})$$
$$=(\lambda-1)O(\delta_m^{\min\{-1+\lambda,-2+\lambda+\varepsilon\}})+O(\delta_m^{\min\{1+\lambda,\lambda+\varepsilon\}})$$
$$+O(\delta_m^{1+\lambda}),$$

and thus by the latter, as $-1+\lambda<1+\lambda$, we have $(\psi_i^{(2)}(t)/2)=\psi_i[t_i,t_{i+1},t_{i+2}]$

$$=(\lambda-1)O(\delta_m^{\min\{-1+\lambda,-2+\lambda+\varepsilon\}})+O(\delta_m^{\min\{1+\lambda,\lambda+\varepsilon\}})$$
$$=\begin{cases}O(\delta_m^{\min\{-1+\lambda,-2+\lambda+\varepsilon\}}), & \text{for } \lambda\in[0,1);\\ O(\delta_m^{\min\{2,1+\varepsilon\}}), & \text{for } \lambda=1;\\ O(\delta_m^{1+\lambda}), & \text{for } t_i=\frac{i}{m},\end{cases} \tag{35}$$

as again $-1+\lambda<1+\lambda$ and $-2+\lambda+\varepsilon\leq\lambda+\varepsilon$. The $O(\delta_m^{1+\lambda})$ asymptotics derived for $\{t_i\}_{i=0}^m$ uniform in (35), comes from the vanishing term $O(\delta_m^{\min\{2,1+\varepsilon\}})$ in (34) (see (11)). Indeed for $t_i=(i/m)$ we have $\delta_m=1/m$, $\phi=id$ and $O(\delta_m^{1+\varepsilon})\equiv 0$ in (5) and $t_{i+2}-t_{i+1}=t_{i+1}-t_i=1/m$. Hence, by (13), we finally obtain for $t\in[t_i,t_{i+2}]$ and for $\lambda\in[0,1]$ the formula (35).

Remark 2. A simple verification shows that formula (33) within the class of merely more-or-less uniform samplings (3) yields for $t\in[t_i,t_{i+2}]$:

$$\psi_i^{(2)}(t)=O(\delta_m^{-2+\lambda}). \tag{36}$$

The asymptotics (36) is independently shown in [11] for (3) under weaker assumption admitting $\gamma\in C^3([0,1])$ instead of $\gamma\in C^4([0,1])$. Visibly, comparison between (35) and (36) gives, for ε-uniform samplings and $\lambda\in[0,1)$, an acceleration of order $\min\{1,\varepsilon\}$ in asymptotics of $O(\psi_i^{(2)}(t))$. In addition, for either $\lambda=1$ and samplings (3) or $\{t_i\}_{i=0}^m$ uniform, formula (33) yields over I_i:

$$\psi_i^{(2)}(t)=O(\delta_m)\quad\text{or}\quad\psi_i^{(2)}(t)=O(\delta_m^{\lambda}), \tag{37}$$

respectively. The first result for this special case in (37) is already proved in [11] for $\gamma\in C^3([0,1])$. Similarly, upon comparing (35) with (37) (for $\lambda=1$) we obtain an extra speed-up of order $\min\{1,\varepsilon\}$ in asymptotics of $O(\psi_i^{(2)}(t))$. On the other hand, once *uniform* sampling is admitted, the last formula from (35) yields faster convergence order $O(\delta_m^{1+\lambda})$ than $O(\delta_m^{\lambda})$ from (37) as shown also by [11], for $\gamma\in C^3([0,1])$. □

The asymptotics of $O(\psi_i^{(1)}(t))$ for ε-uniform samplings (5) by (13), (31) and (35), over I_i reads with $\psi_i^{(1)}(t)$

$$= (t_{i+2}-t_{i+1})^{-1+\lambda}\left(1-\frac{\lambda(t_{i+2}-t_{i+1})^2}{24}\kappa^2(t_{i+1})\right)$$
$$+O((t_{i+2}-t_{i+1})^{2+\lambda})$$
$$+((t-t_i)+(t-t_{i+1}))$$
$$\cdot\begin{cases} O(\delta_m^{\min\{-1+\lambda,-2+\lambda+\varepsilon\}}), & \text{for } \lambda\in[0,1);\\ O(\delta_m^{\min\{2,1+\varepsilon\}}), & \text{for } \lambda=1;\\ O(\delta_m^{1+\lambda}), & \text{for } t_i=\frac{i}{m};\end{cases}$$

$$=\begin{cases} O(\delta_m^{-1+\lambda}), & \text{for } \lambda\in[0,1);\\ 1+O(\delta_m^2), & \text{for } \lambda=1;\\ \delta_m^{-1+\lambda}+O(\delta_m^{1+\lambda}), & \text{for } t_i=\frac{i}{m};\end{cases}$$

$$+\begin{cases} O(\delta_m)O(\delta_m^{\min\{-1+\lambda,-2+\lambda+\varepsilon\}}), & \text{for } \lambda\in[0,1);\\ O(\delta_m)O(\delta_m^{\min\{2,1+\varepsilon\}}), & \text{for } \lambda=1;\\ O(\delta_m)O(\delta_m^{1+\lambda}), & \text{for } t_i=\frac{i}{m};\end{cases}$$

$$=\begin{cases} O(\delta_m^{-1+\lambda})+O(\delta_m^{\min\{\lambda,-1+\lambda+\varepsilon\}}), & \text{for } \lambda\in[0,1);\\ 1+O(\delta_m^2)+O(\delta_m^{\min\{3,2+\varepsilon\}}), & \text{for } \lambda=1;\\ \delta_m^{-1+\lambda}+O(\delta_m^{1+\lambda})+O(\delta_m^{2+\lambda}), & \text{for } t_i=\frac{i}{m};\end{cases}$$

$$=\begin{cases} O(\delta_m^{-1+\lambda}), & \text{for } \lambda\in[0,1);\\ 1+O(\delta_m^2), & \text{for } \lambda=1;\\ \delta_m^{-1+\lambda}+O(\delta_m^{1+\lambda}), & \text{for } t_i=\frac{i}{m};\end{cases}$$

$$=\begin{cases} O(\delta_m^{-1+\lambda}), & \text{for } \lambda\in[0,1);\\ \delta_m^{-1+\lambda}+O(\delta_m^{1+\lambda}), & \text{for } t_i=\frac{i}{m} \text{ or } \lambda=1.\end{cases} \tag{38}$$

The condition (19) forcing ψ_i to be a reparameterization for ε-uniform samplings is later exploited to compare both curves γ and $\hat{\gamma}_2$ defined originally over different domains $[0,1]$ and $[0,\hat{T}]$ (with $\hat{T}=\hat{t}_m$ - see (6)), respectively.

Remark 3. Formula (38) reveals that the asymptotics of $O(\psi_i^{(1)}(t))$ for ε-uniform samplings does depend on ε (contrary to $O(\psi_i^{(2)}(t))$ - see (35)). In addition, if more-or-less uniform sampling (3) is combined with (13), (31) and (33), for $t\in[t_i,t_{i+2}]$ and $\lambda\in[0,1]$, we obtain that $\psi_i^{(1)}(t)$

$$= (t_{i+2}-t_{i+1})^{-1+\lambda}\left(1-\frac{\lambda(t_{i+2}-t_{i+1})^2}{24}\kappa^2(t_{i+1})\right)$$
$$+O\left((t_{i+2}-t_{i+1})^{2+\lambda}\right)$$
$$+((t-t_i)+(t-t_{i+1}))$$
$$\cdot\left(\frac{(t_{i+2}-t_{i+1})^{-1+\lambda}\left(1-\frac{\lambda(t_{i+2}-t_{i+1})^2}{24}\kappa^2(t_i)\right)}{t_{i+2}-t_i}-\frac{(t_{i+1}-t_i)^{-1+\lambda}\left(1-\frac{\lambda(t_{i+1}-t_i)^2}{24}\kappa^2(t_i)\right)}{t_{i+2}-t_i}\right.$$
$$\left.+O\left((t_{i+2}-t_{i+1})^{1+\lambda}\right)+O\left((t_{i+2}-t_{i+1})^{1+\lambda}\right)\right)$$

$$=\begin{cases} O(\delta_m^{-1+\lambda}), & \text{for } \lambda\in[0,1);\\ \delta_m^{\lambda-1}+O(\delta_m^{1+\lambda}), & \text{for } t_i=\frac{i}{m} \text{ or } \lambda=1.\end{cases} \tag{39}$$

Visibly, both asymptotics established for curves $\gamma\in C^4([0,1])$ in either (38) (sampled along (5)) or in (39) (sampled according to (3)) coincide. In addition, the orders of $O(\psi_i^{(1)}(t))$ derived for $\gamma\in C^3([0,1])$ and samplings (3) in [11] are also the same to those specified in (38). Thus, as compared with [11], for estimating $O(\psi_i^{(1)}(t))$ neither raising the smoothness of γ nor restricting samplings $\{t_i\}_{i=0}^m$ to ε-uniformity improves the examined asymptotics for regular $\gamma\in C^3([0,1])$. $\square$

2.4 Step 4: the asymptotic orders of $\hat{\gamma}''_{2,i}(\hat{t})$

We discuss now the asymptotics of $O(\hat{\gamma}''_{2,i}(\hat{t}))$ in terms of δ_m. Similarly to (13), as for each $\hat{t}\in\hat{I}_i=[\hat{t}_i,\hat{t}_{i+2}]$:
$\hat{\gamma}_{2,i}(\hat{t})=\gamma_{2,i}[\hat{t}_i,\hat{t}_{i+1}](\hat{t}-\hat{t}_i)+\hat{\gamma}_{2,i}[\hat{t}_i,\hat{t}_{i+1},\hat{t}_{i+2}](\hat{t}-\hat{t}_i)(\hat{t}-\hat{t}_{i+1})$
we have $\hat{\gamma}''_{2,i}(\hat{t}) = 2\hat{\gamma}_{2,i}[\hat{t}_i,\hat{t}_{i+1},\hat{t}_{i+2}]$ and thus $\hat{\gamma}''_{2,i}(\hat{t})=O(\hat{\gamma}_{2,i}[\hat{t}_i,\hat{t}_{i+1},\hat{t}_{i+2}])$. Since $\hat{\gamma}_{2,i}(\hat{t}_{i+j})=\gamma(t_{i+j})$ (for $j=0,1,2$), by (6) we obtain the following:

$$\hat{\gamma}_{2,i}[\hat{t}_i,\hat{t}_{i+1}]=\frac{\gamma(t_{i+1})-\gamma(t_i)}{\|\gamma(t_{i+1})-\gamma(t_i)\|^{\lambda}}=\frac{\gamma(t_{i+1})-\gamma(t_i)}{(\|\gamma(t_{i+1})-\gamma(t_i)\|^2)^{\frac{\lambda}{2}}}.$$

The latter with (26), (30) and Taylor's expansion gives for $\hat{\gamma}_{2,i}[\hat{t}_i,\hat{t}_{i+1}]=$

$$\frac{(t_{i+1}-t_i)^{1-\lambda}\left(\sum_{k=1}^3\frac{\gamma^{(k)}(t_i)}{k!}(t_{i+1}-t_i)^{k-1}+O\big((t_{i+1}-t_i)^3\big)\right)}{1-\frac{\lambda(t_{i+1}-t_i)^2}{24}\kappa^2(t_i)+O\left((t_{i+1}-t_i)^3\right)}.$$

Again Taylor's expansion about $x_0=0$ applied to the function $(1+x)^{-1}$ with $x=-\frac{\lambda(t_{i+1}-t_i)^2}{24}+O\left((t_{i+1}-t_i)^3\right)$ (separated asymptotically from -1) yields:

$$\frac{1}{1-\frac{\lambda(t_{i+1}-t_i)^2}{24}\kappa^2(t_i)+O((t_{i+1}-t_i)^3)}=1+\frac{\lambda(t_{i+1}-t_i)^2}{24}\kappa^2(t_i)+O((t_{i+1}-t_i)^3).$$

Consequently, over $\hat{I}_i$ we have that $\hat{\gamma}_{2,i}[\hat{t}_i,\hat{t}_{i+1}]$

$$=\left(\sum_{k=1}^3\frac{\gamma^{(k)}(t_i)}{k!}(t_{i+1}-t_i)^{k-1}+O\left((t_{i+1}-t_i)^3\right)\right)\cdot\frac{\left(1+\frac{\lambda(t_{i+1}-t_i)^2}{24}\kappa^2(t_i)+O\left((t_{i+1}-t_i)^3\right)\right)}{(t_{i+1}-t_i)^{\lambda-1}}$$

$$= \left(\sum_{k=1}^{3}\frac{\gamma^{(k)}(t_i)}{k!}(t_{i+1}-t_i)^{k-1}\right)\left(1+\frac{\lambda(t_{i+1}-t_i)^2}{24}\kappa^2(t_i)\right)$$
$$\cdot(t_{i+1}-t_i)^{1-\lambda}+O\left((t_{i+1}-t_i)^{4-\lambda}\right)$$
$$= (t_{i+1}-t_i)^{1-\lambda}\left(\gamma^{(1)}(t_i)+\frac{t_{i+1}-t_i}{2}\gamma^{(2)}(t_i)\right)$$
$$+O\left((t_{i+1}-t_i)^{3-\lambda}\right)+O\left((t_{i+1}-t_i)^{4-\lambda}\right).$$

Hence, as $\gamma^{(1)}(t_{i+1}) = \gamma^{(1)}(t_i) + \gamma^{(2)}(t_i)(t_{i+1}-t_i) + O((t_{i+1}-t_i)^2)$ and $\gamma^{(2)}(t_{i+1}) = \gamma^{(2)}(t_i)+O((t_{i+1}-t_i))$ we have:

$$\hat{\gamma}_{2,i}[\hat{t}_i,\hat{t}_{i+1}]$$
$$= (t_{i+1}-t_i)^{1-\lambda}\left(\gamma^{(1)}(t_i)+\frac{t_{i+1}-t_i}{2}\gamma^{(2)}(t_i)\right)$$
$$+O\left((t_{i+1}-t_i)^{3-\lambda}\right),$$
$$\hat{\gamma}_{2,i}[\hat{t}_{i+1},\hat{t}_{i+2}]$$
$$= (t_{i+2}-t_{i+1})^{1-\lambda}\left(\gamma^{(1)}(t_{i+1})+\frac{t_{i+2}-t_{i+1}}{2}\gamma^{(2)}(t_{i+1})\right)$$
$$+O\left((t_{i+2}-t_{i+1})^{3-\lambda}\right),$$
$$= (t_{i+2}-t_{i+1})^{1-\lambda}\left(\gamma^{(1)}(t_i)+\frac{t_{i+2}+t_{i+1}-2t_i}{2}\gamma^{(2)}(t_i)\right)$$
$$+O(\delta_m^{3-\lambda}). \tag{40}$$

Taking into account that (30) and (31) we arrive at (for $j=0,1$):

$$(\|\gamma(t_{i+j+1})-\gamma(t_{i+j})\|^2)^{\frac{\lambda}{2}}$$
$$= \frac{\left(1-\frac{\lambda(t_{i+j+1}-t_{i+j})^2}{24}\kappa^2(t_{i+j})+O((t_{i+j+1}-t_{i+j})^3)\right)}{(t_{i+j+1}-t_{i+j})^{-\lambda}}. \tag{41}$$

So, by (6), (40) and (41) for merely more-or-less uniform samplings (3) (and hence for each ε-uniform samplings) the second divided difference, upon introducing the substitutions:

$$A = \gamma^{(1)}(t_i)+\frac{t_{i+2}+t_{i+1}-2t_i}{2}\gamma^{(2)}(t_i),$$
$$B = \gamma^{(1)}(t_i)+\frac{t_{i+1}-t_i}{2}\gamma^{(2)}(t_i),$$

the second divided difference $\|\hat{\gamma}_{2,i}[\hat{t}_i,\hat{t}_{i+1},\hat{t}_{i+2}]\|$ amounts to:

$$= \frac{\|\hat{\gamma}_{2,i}[\hat{t}_{i+1},\hat{t}_{i+2}]-\hat{\gamma}_{2,i}[\hat{t}_i,\hat{t}_{i+1}]\|}{(\hat{t}_{i+2}-\hat{t}_{i+1})+(\hat{t}_{i+1}-\hat{t}_i)}$$
$$\leq \frac{\|(t_{i+2}-t_{i+1})^{1-\lambda}\cdot A+O(\delta_m^{3-\lambda})\|}{\|\gamma(t_{i+1})-\gamma(t_i)\|^{\lambda}+\|\gamma(t_{i+2})-\gamma(t_{i+1})\|^{\lambda}}$$
$$+\frac{\|(t_{i+1}-t_i)^{1-\lambda}\cdot B+O\left((t_{i+1}-t_i)^{3-\lambda}\right)\|}{\|\gamma(t_{i+1})-\gamma(t_i)\|^{\lambda}+\|\gamma(t_{i+2})-\gamma(t_{i+1})\|^{\lambda}}$$
$$\leq \frac{\|(t_{i+2}-t_{i+1})^{1-\lambda}\cdot A+O(\delta_m^{3-\lambda})\|}{(\|\gamma(t_{i+2})-\gamma(t_{i+1})\|^2)^{\frac{\lambda}{2}}}$$
$$+\frac{\|(t_{i+1}-t_i)^{1-\lambda}\cdot B+O\left((t_{i+1}-t_i)^{3-\lambda}\right)\|}{(\|\gamma(t_{i+1})-\gamma(t_i)\|^2)^{\frac{\lambda}{2}}}$$
$$= \frac{\|(t_{i+2}-t_{i+1})^{1-2\lambda}\cdot A+O(\delta_m^{3-2\lambda})\|}{1-\frac{\lambda(t_{i+2}-t_{i+1})^2}{24}\kappa^2(t_{i+1})+O(\delta_m^3)}$$
$$+\frac{\|(t_{i+1}-t_i)^{1-2\lambda}\cdot B+O\left((t_{i+1}-t_i)^{3-2\lambda}\right)\|}{1-\frac{\lambda(t_{i+1}-t_i)^2}{24}\kappa^2(t_i)+O(\delta_m^3)}. \tag{42}$$

Taylor's expansion applied to $(1+x)^{-1}$ about $x_0=0$ yields (for $j=0,1$):

$$\left(1-\frac{\lambda(t_{i+j+1}-t_{i+j})^2}{24}\kappa^2(t_{i+j})+O(\delta_m^3)\right)^{-1}$$
$$= 1+\frac{\lambda(t_{i+j+1}-t_{i+j})^2}{24}\kappa^2(t_{i+j})+O(\delta_m^3),$$

and hence by (42):

$$\hat{\gamma}_{2,i}[\hat{t}_i,\hat{t}_{i+1},t_{i+2}] = O(\delta_m^{1-2\lambda})+O(\delta_m^{2-2\lambda})$$
$$= O(\delta_m^{1-2\lambda}). \tag{43}$$

In the special case when $\{t_i\}_{i=0}^m$ is *uniform*, the formulas (40) and (41) (with $t_{i+1}-t_i = t_{i+2}-t_{i+1} = \delta_m = (1/m)$) give:

$$\hat{\gamma}_{2,i}[\hat{t}_i,\hat{t}_{i+1},\hat{t}_{i+2}]$$
$$= \frac{\delta_m^{1-\lambda}\left(\frac{3\delta_m}{2}\gamma^{(2)}(t_i)-\frac{\delta_m}{2}\gamma^{(2)}(t_i)\right)+O(\delta_m^{3-\lambda})}{\delta_m^{\lambda}(1+O(\delta_m^2))}$$
$$= \frac{\delta_m^{2-2\lambda}\gamma^{(2)}(t_i)+O(\delta_m^{3-2\lambda})}{1+O(\delta_m^2)}$$
$$= O(\delta_m^{2-2\lambda}). \tag{44}$$

Such accelerated convergence order for uniform samplings (as compared with (43)) can also be found in [11] for curves $\gamma\in C^3([0,1])$.

Finally, for another special case i.e. $\lambda=1$ and samplings merely admissible (1), by (40), (41), $|(t_{i+j+1}-t_{i+j})/(t_{i+2}-t_i)|\leq 1$ (with $j=0,1$) and $\gamma^{(1)}(t_{i+1}) = \gamma^{(1)}(t_i)+O(t_{i+1}-t_i)$, upon substituting (for $k=0,1$):

$$C(k) = (t_{i+k+1}-t_{i+k})(1+O((t_{i+k+1}-t_{i+k})^2)$$

the divided differences $\hat{\gamma}_{2,i}[\hat{t}_i,\hat{t}_{i+1},\hat{t}_{i+2}]$

$$= \frac{\gamma^{(1)}(t_{i+1})+\frac{t_{i+2}-t_{i+1}}{2}\gamma^{(2)}(t_{i+1})-\gamma^{(1)}(t_i)-\frac{t_{i+1}-t_i}{2}\gamma^{(2)}(t_i)}{C(1)+C(0)}$$
$$+\frac{O\left((t_{i+2}-t_{i+1})^2\right)+O\left((t_{i+1}-t_i)^2\right)}{C(1)+C(0))}$$
$$= \frac{O(t_{i+1}-t_i)+O(t_{i+2}-t_{i+1})}{(t_{i+2}-t_i)+O\left((t_{i+2}-t_{i+1})^3\right)+O\left((t_{i+1}-t_i)^3\right)}$$

$$+\frac{O\left((t_{i+1}-t_i)^2\right)+O\left((t_{i+2}-t_{i+1})^2\right)}{(t_{i+2}-t_i)+O\left((t_{i+2}-t_{i+1})^3\right)+O\left((t_{i+1}-t_i)^3\right)}$$
$$=\frac{O(1)+O(t_{i+1}-t_i)+O(t_{i+2}-t_{i+1})}{1+O\left((t_{i+2}-t_{i+1})^2\right)+O\left((t_{i+1}-t_i)^2\right)}$$
$$=O(1). \tag{45}$$

Here we use Taylor's expansion with $x_0=0$ applied to $(1+x)^{-1}$ at $x=O\left((t_{i+1}-t_i)^2\right)+O\left((t_{i+2}-t_{i+1})^2\right)$. Note that (45) coincides with (44) once $\lambda=1$. Thus a single formula (44) covers both $\lambda=1$ or uniform samplings. This result is the same as before in [11] for curves merely $\gamma\in C^3([0,1])$. Hence collating (43), (44) and (45) for $\hat{t}\in[\hat{t}_i,\hat{t}_{i+2}]$ (with each unique $t=\psi_i^{-1}(\hat{t})\in[t_i,t_{i+2}]$ since ψ_i is a reparameterization as shown in *Step 1*) the following holds:

$$\gamma''_{2,i}(\hat{t})=\begin{cases}O(\delta_m^{1-2\lambda}), & \text{for } \lambda\in[0,1);\\ O(\delta_m^{2-2\lambda}), & \text{for } t_i=\frac{i}{m} \text{ or } \lambda=1.\end{cases} \tag{46}$$

We exploit now (5) of $\{t_i\}_{i=0}^m$. An extra acceleration is achievable for the asymptotics of $O(\gamma''_{2,i})$ once both formulas (40) and (41) derived for $\gamma\in C^4([0,1])$ are considered with more care.

Remark 4. The analysis so-far indicates that an increase of smoothness in γ from $C^3([0,1])$ to $C^4([0,1])$ does not contribute on its own (as compared with [11]) to faster orders for $O(\gamma''^{(2)}_{2,i})$ than for more-or-less uniform samplings. Indeed a trajectory estimation for samplings (3) and regular curves $\gamma\in C^4([0,1])$ by (25), (36), (37), (39) and (46) reads as $f(t)$

$$=O(\delta_m^3)$$
$$\cdot\begin{cases}O(\delta_m^{1-2\lambda})O(\delta_m^{-1+\lambda})O(\delta_m^{-2+\lambda}), & \lambda\in[0,1);\\ O(\delta_m^{2-2\lambda})(\delta_m^{-1+\lambda}+O(\delta_m^{1+\lambda}))O(\delta_m^{\lambda}), & t_i=\frac{i}{m};\\ O(\delta_m^{2-2\lambda})(\delta_m^{-1+\lambda}+O(\delta_m^{1+\lambda}))O(\delta_m^{\lambda}), & \lambda=1;\end{cases}$$
$$+O(\delta_m^3)\begin{cases}O(1), & \text{for } \lambda\in[0,1);\\ O(1), & \text{for } t_i=\frac{i}{m} \text{ or } \lambda=1;\end{cases}$$
$$=O(\delta_m^3)\begin{cases}O(\delta_m^{-2})+O(1), & \text{for } \lambda\in[0,1);\\ O(\delta_m)+O(1), & \text{for } t_i=\frac{i}{m} \text{ or } \lambda=1;\end{cases}$$
$$=\begin{cases}O(\delta_m), & \text{for } \lambda\in[0,1);\\ O(\delta_m^3), & \text{for } t_i=\frac{i}{m} \text{ or } \lambda=1.\end{cases} \tag{47}$$

over $[0,1]$. □

We prove now that for ε-uniform samplings the asymptotics in (46), as in (35) (and hence also in (47)) can be accelerated. In fact, to improve the estimate of $\gamma''_2(\hat{t})$ we argue as in (42). Indeed, by (41), ε-uniformity (5), (11) and by Taylor's expansion applied to $(1+x)^{\lambda}$ we arrive at $\|\gamma(t_{i+2})-\gamma(t_{i+1})\|^{\lambda}$

$$=(t_{i+2}-t_{i+1})^{\lambda}$$
$$\cdot\left(1-\frac{\lambda}{24}k^2(t_{i+1})(t_{i+2}-t_{i+1})^2+O((t_{i+2}-t_{i+1})^3)\right)$$
$$=\left((t_{i+1}-t_i)+O(\delta_m^{\min\{2,1+\varepsilon\}})\right)^{\lambda}$$
$$\cdot\left(1-\frac{\lambda}{24}k^2(t_{i+1})(t_{i+2}-t_{i+1})^2+O((t_{i+2}-t_{i+1})^3)\right)$$
$$=(t_{i+1}-t_i)^{\lambda}\cdot\left(1+O(\delta_m^{\min\{1,\varepsilon\}})\right)^{\lambda}$$
$$\cdot\left(1-\frac{\lambda}{24}k^2(t_{i+1})(t_{i+2}-t_{i+1})^2\right)$$
$$+O((t_{i+2}-t_{i+1})^{3+\lambda})$$
$$=(t_{i+1}-t_i)^{\lambda}\cdot\left(1+O(\delta_m^{\min\{1,\varepsilon\}})\right)$$
$$\cdot\left(1-\frac{\lambda}{24}k^2(t_{i+1})(t_{i+2}-t_{i+1})^2\right)$$
$$+O((t_{i+2}-t_{i+1})^{3+\lambda}). \tag{48}$$

Similarly $\|\gamma(t_{i+1})-\gamma(t_i)\|^{\lambda}$

$$=(t_{i+1}-t_i)^{\lambda}$$
$$\cdot\left(1-\frac{\lambda}{24}k^2(t_i)(t_{i+1}-t_i)^2\right)+O((t_{i+1}-t_i)^{3+\lambda}). \tag{49}$$

Coupling formula (48) with (49) leads to $\left(\|\gamma(t_{i+2})-\gamma(t_{i+1})\|^{\lambda}+\|\gamma(t_{i+1})-\gamma(t_i)\|^{\lambda}\right)^{-1}$

$$=\frac{1}{(t_{i+1}-t_i)^{\lambda}(2+O(\delta_m^{\min\{1,\varepsilon\}}))}$$
$$=(t_{i+1}-t_i)^{-\lambda}(2+O(\delta_m^{\min\{1,\varepsilon\}})), \tag{50}$$

where Taylor's expansion is applied to $(2+x)^{-1}$ at $x=O(\delta_m^{\min\{1,\varepsilon\}})$. Furthermore by (11), (40), (41), (50) combined with (3), $\gamma^{(1)}(t_{i+1})=\gamma^{(1)}(t_i)+O(\delta_m)$ and Taylor's expansion $(1+x)^{1-\lambda}$ we obtain for the divided difference $\hat{\gamma}_2[\hat{t}_i,\hat{t}_{i+1},\hat{t}_{i+2}]$

$$=\frac{(t_{i+2}-t_{i+1})^{1-\lambda}\left(\gamma^{(1)}(t_{i+1})+O(\delta_m)\right)}{\|\gamma(t_{i+1})-\gamma(t_i)\|^{\lambda}+\|\gamma(t_{i+2})-\gamma(t_{i+1})\|^{\lambda}}$$
$$-\frac{(t_{i+1}-t_i)^{1-\lambda}\left(\gamma^{(1)}(t_i)+O(\delta_m)\right)}{\|\gamma(t_{i+1})-\gamma(t_i)\|^{\lambda}+\|\gamma(t_{i+2})-\gamma(t_{i+1})\|^{\lambda}}+O(\delta_m^{3-2\lambda})$$
$$=\frac{(t_{i+2}-t_{i+1})^{1-\lambda}\left(\gamma^{(1)}(t_i)+O(\delta_m)\right)}{\|\gamma(t_{i+1})-\gamma(t_i)\|^{\lambda}+\|\gamma(t_{i+2})-\gamma(t_{i+1})\|^{\lambda}}$$
$$-\frac{(t_{i+1}-t_i)^{1-\lambda}\left(\gamma^{(1)}(t_i)+O(\delta_m)\right)}{\|\gamma(t_{i+1})-\gamma(t_i)\|^{\lambda}+\|\gamma(t_{i+2})-\gamma(t_{i+1})\|^{\lambda}}+O(\delta_m^{3-2\lambda})$$
$$=\frac{\left((t_{i+1}-t_i)+O(\delta_m^{\min\{2,1+\varepsilon\}})\right)^{1-\lambda}\gamma^{(1)}(t_i)}{\|\gamma(t_{i+1})-\gamma(t_i)\|^{\lambda}+\|\gamma(t_{i+2})-\gamma(t_{i+1})\|^{\lambda}}$$

$$-\frac{(t_{i+1}-t_i)^{1-\lambda}\gamma^{(1)}(t_i)}{\|\gamma(t_{i+1})-\gamma(t_i)\|^\lambda+\|\gamma(t_{i+2})-\gamma(t_{i+1})\|^\lambda}+O(\delta_m^{2-2\lambda})$$
$$+O(\delta_m^{3-2\lambda})$$
$$=\frac{(t_{i+1}-t_i)^{1-\lambda}\left(1+O(\delta^{\min\{1,\varepsilon\}})\right)^{1-\lambda}\gamma^{(1)}(t_i)}{\|\gamma(t_{i+1})-\gamma(t_i)\|^\lambda+\|\gamma(t_{i+2})-\gamma(t_{i+1})\|^\lambda}$$
$$-\frac{(t_{i+1}-t_i)^{1-\lambda}\gamma^{(1)}(t_i)}{\|\gamma(t_{i+1})-\gamma(t_i)\|^\lambda+\|\gamma(t_{i+2})-\gamma(t_{i+1})\|^\lambda}+O(\delta_m^{2-2\lambda})$$
$$=\frac{(t_{i+1}-t_i)^{1-\lambda}\left(1+(1-\lambda)O(\delta_m^{\min\{1,\varepsilon\}})\right)\gamma^{(1)}(t_i)}{\|\gamma(t_{i+1})-\gamma(t_i)\|^\lambda+\|\gamma(t_{i+2})-\gamma(t_{i+1})\|^\lambda}$$
$$-\frac{(t_{i+1}-t_i)^{1-\lambda}\gamma^{(1)}(t_i)}{\|\gamma(t_{i+1})-\gamma(t_i)\|^\lambda+\|\gamma(t_{i+2})-\gamma(t_{i+1})\|^\lambda}+O(\delta_m^{2-2\lambda})$$
$$=\frac{(1-\lambda)O(\delta_m^{\min\{2-\lambda,1+\varepsilon-\lambda\}})}{\|\gamma(t_{i+1})-\gamma(t_i)\|^\lambda+\|\gamma(t_{i+2})-\gamma(t_{i+1})\|^\lambda}+O(\delta_m^{2-2\lambda})$$
$$=(1-\lambda)O(\delta_m^{\min\{2-2\lambda,1+\varepsilon-2\lambda\}})+O(\delta_m^{2-2\lambda})$$
$$=O(\delta_m^{\min\{2-2\lambda,1+\varepsilon-2\lambda\}}). \tag{51}$$

Note that if $\lambda=1$ then (51) yields $\gamma''_{2,i}(\hat{t})=O(1)$ which coincides with (46). Similarly, if in (51), uniform sampling is used (i.e when term $O(\delta_m^{\min\{2,1+\varepsilon\}})$ in (5) and (11) vanishes), evidently we have $\gamma''_{2,i}(\hat{t})=O(\delta_m^{2-2\lambda})$ which again is already claimed by (46). In summary, over $\hat{I}_i$, for $\lambda\in[0,1]$ and ε-uniform samplings the following holds:

$$\hat{\gamma}''_2(\hat{t})=\begin{cases}O(\delta_m^{\min\{2-2\lambda,1+\varepsilon-2\lambda\}}), & \text{for } \lambda\in[0,1);\\ O(\delta_m^{2-2\lambda}), & \text{for } t_i=\frac{i}{m} \text{ or } \lambda=1.\end{cases} \tag{52}$$

Formula (52) as compared with (46) yields, for all $\lambda\in[0,1)$ an acceleration by either ε for $0<\varepsilon\le 1$ or by 1 for $\varepsilon\ge 1$. (In addition, the case $\lambda=1$ relaxes the assumption concerning $\{t_i\}_{i=0}^m$ to form merely admissible samplings (1).)

2.5 *Step 5: asymptotics for trajectory estimation*

We pass now to the final stage of the asymptotic estimate for γ approximation by interpolant $\hat{\gamma}_2$. It is essential to observe that both curves γ and $\hat{\gamma}_2$ are originally defined over different domains i.e. over $[0,1]$ and $[0,\hat{T}]$, respectively. The piecewise-quadratic $\psi:[0,1]\to[0,\hat{T}]$ (a track-sum of $\psi_i:[t_i,t_{i+2}]\to[\hat{t}_i,\hat{t}_{i+2}]$) applied here to compare γ and $\hat{\gamma}_2\circ\psi$, as demonstrated in *Step 1* forms a genuine reparameterization of $[0,1]$ into $[0,\hat{T}]$ for arbitrary ε-uniform samplings (5). The latter may not be the case for the general class of more-or-less uniform samplings (3) (see [11]).

Using (25), (35), (38) and (52) with ε-uniformity yields for $\lambda\in[0,1]$ the following approximation orders in trajectory estimation error over each I_i reading as $f_i(t)$

$$=O(\delta_m^3)O(1)+$$
$$\begin{cases}O(\delta_m^3)O(\delta_m^{\min\{2-2\lambda,1+\varepsilon-2\lambda\}}), & \text{for } \lambda\in[0,1);\\ O(\delta_m^3)O(1), & \text{for } \lambda=1;\\ O(\delta_m^3)O(\delta_m^{2-2\lambda}), & \text{for } t_i=\frac{i}{m};\end{cases}$$
$$\cdot\begin{cases}O(\delta_m^{-1+\lambda})O(\delta_m^{\min\{-1+\lambda,-2+\lambda+\varepsilon\}}), & \text{for } \lambda\in[0,1);\\ \left(1+O(\delta_m^2)\right)O(\delta_m^{\min\{2,1+\varepsilon\}}), & \text{for } \lambda=1;\\ \left(\delta_m^{-1+\lambda}+O(\delta_m^{1+\lambda})\right)O(\delta_m^{1+\lambda}), & \text{for } t_i=\frac{i}{m};\end{cases}$$
$$=O(\delta_m^3)+$$
$$\begin{cases}O(\delta_m^{\min\{5-2\lambda,4+\varepsilon-2\lambda\}+\min\{-2+2\lambda,-3+2\lambda+\varepsilon\}}), & \lambda\in[0,1);\\ O(\delta_m^{\min\{5,4+\varepsilon\}}), & \lambda=1;\\ O(\delta_m^5), & t_i=\frac{i}{m}.\end{cases} \tag{53}$$

We re-emphasized here that for $\lambda=1$ the constraint on samplings $\{t_i\}_{i=0}^m$ in (53) are the loosest, i.e. only condition (1) is imposed. Upon noting that both inequalities $5-2\lambda\le 4+\varepsilon-2\lambda$ and $2\lambda-2\le 2\lambda+\varepsilon-3$ hold if and only if $\varepsilon\ge 1$ formula (53) reduces to:

$$f(t)=O(\delta_m^3),$$
$$+\begin{cases}O(\delta_m^{1+2\varepsilon}), & \text{for } 0<\varepsilon\le 1\ \&\ \lambda\in[0,1);\\ O(\delta_m^3), & \text{for } \varepsilon>1\ \&\ \lambda\in[0,1);\\ O(\delta_m^{\min\{5,4+\varepsilon\}}), & \text{for } \lambda=1;\\ O(\delta_m^5), & \text{for } t_i=\frac{i}{m};\end{cases}$$
$$=\begin{cases}O(\delta_m^{\min\{3,1+2\varepsilon\}}), & \text{for } \lambda\in[0,1);\\ O(\delta_m^3), & \text{for } t_i=\frac{i}{m} \text{ or } \lambda=1.\end{cases} \tag{54}$$

The above asymptotics applies over each sub-interval I_i. As the bounds involved are independent from I_i, the formula (8) holds over $[0,1]$. Consequently, the proof of Th. 4 is complete. $\square$

Remark 5. For (8) it suffices to take $\{\hat{t}_i\}_{i=0}^m$ instead of the re-normalized $\{\tilde{t}_i\}_{i=0}^m$ (see (6)). The linear mapping $\theta_i:[\hat{t}_i,\hat{t}_{i+2}]\to[\tilde{t}_i,\tilde{t}_{i+2}]$, where $\tilde{t}=\theta_i(\hat{t})=\hat{t}/\hat{T}$ satisfies $\theta_i(\hat{t}_{i+j})=\tilde{t}_{i+j}$, for $j=0,1,2$. A quadratic $\tilde{\gamma}_{2,i}:[\tilde{t}_i,\tilde{t}_{i+2}]\to E^n$ which fulfills $\tilde{\gamma}_{2,i}(\tilde{t}_{i+j})=q_{i+j}$ corresponds to the quadratic $\hat{\gamma}_{2,i}:[\hat{t}_i,\hat{t}_{i+2}]\to E^n$ satisfying $\hat{\gamma}_{2,i}(\hat{t}_{i+j})=q_{i+j}$, where $\tilde{\gamma}_{2,i}=\hat{\gamma}_{2,i}\circ\theta_i^{-1}$. Let $\tilde{\psi}_i:[t_i,t_{i+2}]\to[\tilde{t}_i,\tilde{t}_{i+2}]$ is a quadratic satisfying $\tilde{\psi}_i(t_{i+j})=\tilde{t}_{i+j}$, for $j=0,1,2$. By linearity of θ_i and uniqueness of Lagrange interpolant we also have $\tilde{\psi}_i=\theta_i\circ\psi_i$. Hence $f(t)=(\hat{\gamma}_{2,i}\circ\psi_i)(t)-\gamma(t)=(\hat{\gamma}_{2,i}\circ\theta_i^{-1}\circ\theta_i\circ\psi_i)(t)-\gamma(t)=(\tilde{\gamma}_{2,i}\circ\tilde{\psi}_i)(t)-\gamma(t)$. Also $\tilde{\psi}_i$ is asymptotically a reparameterization since $\dot{\psi}_i>0$, for sufficiently large m (see Step 1 in Th. 4). Thus the asymptotics derived in (54) prevails equally for $(\tilde{\gamma}_{2,i}\circ\tilde{\psi}_i)(t)-\gamma(t)$. The shift in $\hat{t}\in[0,\hat{t}_{i+2}-\hat{t}_i]$ used in

Step 1 does not change the asymptotics in (54) as the curve $\hat{\gamma}_{2,i,s}(\hat{t}) = \hat{\gamma}_{2,i}(\hat{t}-\hat{t}_i)$ satisfies $\hat{\gamma}''_{2,i,s}(\hat{t}) = \hat{\gamma}''_{2,i}(\hat{t})$. $\square$

Note that for ε-uniform samplings Th. 4 extends Th. 2 (claimed for $\lambda = 0$) to $\lambda \in [0,1)$. The estimates established in Th. 4 are *sharp* (as shown in Ex. 2). Consequently by Th. 4 any increment within the interval $\lambda \in [0,1)$ does not bring a further extra convergence acceleration (for ε-uniform samplings) different than 2ε established earlier for $\lambda = 0$ in Th. 2. Moreover, the bigger ε in (5) is, the closer, modulo a diffeomorphism ϕ, the sampling $\{t_i\}_{i=0}^m$ approaches a uniform sampling. Indeed, this is manifested in (54), where cubic convergence order $O(\delta_m^3)$ established for $t_i = i/m$ is attained with $\varepsilon \geq 1$. The case when $\lambda = 1$ (see Th. 3) is also covered by Th. 4.

The next example confirms analytically the sharpness of Th. 4. Recall that sharpness for samplings (3) with $\lambda \in [0,1]$ or for $\lambda = 1$ and samplings (1) is already demonstrated in [11]. We pass now to the case when $\lambda \in [0,1)$ and ε-uniform samplings are admitted.

Example 2. Consider the ε-uniform sampling such that for some knots $\{t_i, t_{i+1}, t_{i+2}\}$ (with $t_0 = 0$):

$$t_{i+1} - t_i = \hat{\delta}_m(1+\hat{\delta}_m^{\varepsilon}), \qquad t_{i+2} - t_{i+1} = \hat{\delta}_m(1-\hat{\delta}_m^{\varepsilon}), \quad (55)$$

where $\hat{\delta}_m = 1/m$. Note that here $\delta_m = \hat{\delta}_m(1+\hat{\delta}_m^{\varepsilon})$ and $\phi = id$ (see (5)). The curve under consideration (*a straight line*) is defined as $\gamma_l(t) = t\mathbf{v}$, where $\|v\| = 1$ and $t \in [0,1]$.

a) For sharpness of (8) (with $\varepsilon \in (0,1]$) it suffices to show that, over I_i we have:

$$f_l(t) = (\hat{\gamma}_2 \circ \psi_i)(t) - \gamma_l(t) = \sigma\delta_m^{1+2\varepsilon} + O(\delta_m^{1+2\varepsilon+\kappa}), \quad (56)$$

for some $\kappa > 0$ and vector $\sigma = (\sigma_1, \sigma_2) \neq \mathbf{0} \in E^2$. Note that the second expression in (56) is a vector in E^2. Since $\hat{\delta}_m^{\rho} = \delta_m^{\rho}(1+\hat{\delta}_m^{\varepsilon})^{-\rho}$ by the Binomial Th. $\hat{\delta}_m^{\rho} = \delta_m^{\rho}(1+O(\hat{\delta}_m^{\varepsilon}))$ and as $\hat{\delta}_m < \delta_m$ we have $\hat{\delta}_m^{\rho} = \delta_m^{\rho}(1+O(\delta_m^{\varepsilon}))$. Thus to justify (56) it is sufficient to substitute δ_m with $\hat{\delta}_m$. It is also enough to prove (56) for some $\bar{t} \in [t_i, t_{i+2}]$. We set here $\bar{t} = (t_i + t_{i+2})/2$. The proof of Lemma 1 yields:

$$f_l(\bar{t}) = (\bar{t}-t_i)(\bar{t}-t_{i+1})(\bar{t}-t_{i+2}) \cdot \int_{[0,1]^3} f_l'''(\eta(\bar{t}))u^2u_1 du du_1 du_2, \quad (57)$$

for the function $\eta(\bar{t}) = \big((\bar{t}u_2 + (1-u_2)t_{i+2})u_1 + (1-u_1)t_{i+1}\big)u + (1-u)t_i$ and where the third derivative of f_l is taken over $\eta(t)$. Furthermore by the Chain Rule, (13) and $\gamma_l'''(t) \equiv \mathbf{0}$ we obtain that:

$$\begin{aligned}
&f_l'''(\eta(\bar{t}))\\
&= 3\hat{\gamma}''_{2,i}(\psi_i(\eta(\bar{t})))\psi_i^{(1)}(\eta(\bar{t}))\psi_i^{(2)}(\eta(\bar{t}))\\
&= 12\hat{\gamma}_{2,i}[\hat{t}_i,\hat{t}_{i+1},\hat{t}_{i+2}]\psi_i[t_i,t_{i+1},t_{i+2}]\\
&\cdot\Big(\psi_i[t_i,t_{i+1}] + (2\eta(\bar{t}) - t_i - t_{i+1})\psi_i[t_i,t_{i+1},t_{i+2}]\Big) \quad (58)
\end{aligned}$$

and that by (55) the following holds:

$$\begin{aligned}
&(\bar{t}-t_i)(\bar{t}-t_{i+1})(\bar{t}-t_{i+2})\\
&= (1/8)(t_{i+1}-t_i)^2\left((t_{i+2}-t_i)+(t_{i+1}-t_i)\right)\\
&= (1/8)\hat{\delta}_m^3(1+\hat{\delta}_m^{\varepsilon})^2(3-\hat{\delta}_m^{\varepsilon}). \quad (59)
\end{aligned}$$

Since $\int_{[0,1]^3} u^2u_1 du du_1 du_2 = 1/6$ formula (57) combined with (58) and (59) yields $f_l(\bar{t}) =$

$$\begin{aligned}
&(3/2)\hat{\delta}_m^2(1+\hat{\delta}_m^{\varepsilon})^2(3\hat{\delta}_m - \hat{\delta}_m^{1+\varepsilon})\\
&\cdot\hat{\gamma}_{2,i}[\hat{t}_i,\hat{t}_{i+1},\hat{t}_{i+2}]\psi_i[t_i,t_{i+1},t_{i+2}]\\
&\cdot\Big((1/6)\psi_i[t_i,t_{i+1}]\\
&+\int_{[0,1]^3}(2\eta(\bar{t})-t_i-t_{i+1})\psi_i[t_i,t_{i+1},t_{i+2}]u^2u_1 du du_1 du_2\Big). \quad (60)
\end{aligned}$$

b) We determine now the asymptotics of the first component $f_{l1}(\bar{t})$ of (60) (assume here the decomposition $f_l(\bar{t}) = f_{l1}(\bar{t}) + f_{l2}(\bar{t})$). Combining (6) and (55) with the Binomial Th.:

$$\begin{aligned}
&\psi_i[t_i,t_{i+1}]\\
&= \frac{\|(t_{i+1}-t_i)\mathbf{v}\|^{\lambda}}{\hat{\delta}_m(1+\hat{\delta}_m^{\varepsilon})}\\
&= \hat{\delta}_m^{-1+\lambda}(1+\hat{\delta}_m^{\varepsilon})^{-1+\lambda}\\
&= \hat{\delta}_m^{-1+\lambda}\cdot\Big(1+(\lambda-1)\hat{\delta}_m^{\varepsilon}+\frac{(\lambda-1)(\lambda-2)}{2}\hat{\delta}_m^{2\varepsilon}\\
&\qquad +(\lambda-1)O(\hat{\delta}_m^{3\varepsilon})\Big)\\
&= \hat{\delta}_m^{-1+\lambda}+(\lambda-1)\hat{\delta}_m^{-1+\varepsilon+\lambda}+(\lambda-1)O(\hat{\delta}_m^{-1+2\varepsilon+\lambda}),\\
&\psi_i[t_{i+1},t_{i+2}]\\
&= \hat{\delta}_m^{-1+\lambda}(1-\hat{\delta}_m^{\varepsilon})^{-1+\lambda}\\
&= \hat{\delta}_m^{-\lambda+1}\cdot\Big(1-(\lambda-1)\hat{\delta}_m^{\varepsilon}+\frac{(\lambda-1)(\lambda-2)}{2}\hat{\delta}_m^{2\varepsilon}\\
&\qquad +(\lambda-1)O(\hat{\delta}_m^{3\varepsilon})\Big)\\
&= \hat{\delta}_m^{-1+\lambda}-(\lambda-1)\hat{\delta}_m^{-1+\varepsilon+\lambda}+(\lambda-1)O(\hat{\delta}_m^{-1+2\varepsilon+\lambda}). \quad (61)
\end{aligned}$$

Therefore, by (55) and (61) we have:

$$\begin{aligned}
&\psi_i[t_i,t_{i+1},t_{i+2}]\\
&= \frac{\hat{\delta}_m^{-1+\lambda}(1-\hat{\delta}_m^{\varepsilon})^{-1+\lambda} - \hat{\delta}_m^{-1+\lambda}(1+\hat{\delta}_m^{\varepsilon})^{-1+\lambda}}{2\hat{\delta}_m}\\
&= (1-\lambda)\hat{\delta}_m^{-2+\varepsilon+\lambda}+(1-\lambda)O(\hat{\delta}_m^{-2+3\varepsilon+\lambda})
\end{aligned}$$

$$= \hat{\delta}_m^{-2+\varepsilon+\lambda}\left((1-\lambda)+(1-\lambda)O(\hat{\delta}_m^{2\varepsilon})\right). \tag{62}$$

The divided differences for $\hat{\gamma}_{2,i}$ upon using again the Binomial Th. read as:

$$\hat{\gamma}_{2,i}[\hat{t}_i,\hat{t}_{i+1},\hat{t}_{i+2}]$$
$$= \frac{\frac{(t_{i+2}-t_{i+1})\mathbf{v}}{(t_{i+2}-t_{i+1})^{\lambda}} - \frac{(t_{i+1}-t_i)\mathbf{v}}{(t_{i+1}-t_i)^{\lambda}}}{\left(\hat{\delta}_m(1-\hat{\delta}_m^{\varepsilon})\right)^{\lambda}\|\mathbf{v}\|^{\lambda}+\left(\hat{\delta}_m(1+\hat{\delta}_m^{\varepsilon})\right)^{\lambda}\|\mathbf{v}\|^{\lambda}}$$
$$= \frac{(\hat{\delta}_m(1-\hat{\delta}_m^{\varepsilon}))^{1-\lambda}\mathbf{v}-(\hat{\delta}_m(1+\hat{\delta}_m^{\varepsilon}))^{1-\lambda}\mathbf{v}}{(\hat{\delta}_m(1-\hat{\delta}_m^{\varepsilon}))^{\lambda}+(\hat{\delta}_m(1+\hat{\delta}_m^{\varepsilon}))^{\lambda}}$$
$$= \frac{\hat{\delta}_m^{1-2\lambda}\left(2(\lambda-1)\hat{\delta}_m^{\varepsilon}+(\lambda-1)O(\hat{\delta}_m^{3\varepsilon})\right)}{2+\lambda(\lambda-1)O(\hat{\delta}_m^{2\varepsilon})}\mathbf{v}$$
$$= \hat{\delta}_m^{1-2\lambda+\varepsilon}\left((\lambda-1)+(\lambda-1)O(\hat{\delta}_m^{2\varepsilon})\right)\mathbf{v} \tag{63}$$

as $\left(1+\lambda(\lambda-1)O(\hat{\delta}_m^{2\varepsilon})\right)^{-1} = 1+O(\hat{\delta}_m^{2\varepsilon})$. Therefore by (61), (62), (63), the first expression $f_{l1}(\bar{t})$ in (60) satisfies:

$$f_{l1}(\bar{t}) = (1/4)\hat{\delta}_m^3(1+2\hat{\delta}_m^{\varepsilon}+\hat{\delta}_m^{2\varepsilon})(3-\hat{\delta}_m^{\varepsilon})\hat{\delta}_m^{-1+\lambda}$$
$$\cdot\left(1+(\lambda-1)\hat{\delta}_m^{\varepsilon}+(\lambda-1)O(\hat{\delta}_m^{2\varepsilon})\right)\hat{\delta}_m^{-2+\varepsilon+\lambda}$$
$$\cdot\left((1-\lambda)+(1-\lambda)O(\hat{\delta}_m^{2\varepsilon})\right)\hat{\delta}_m^{1-2\lambda+\varepsilon}$$
$$\cdot\left((\lambda-1)+(\lambda-1)O(\hat{\delta}_m^{2\varepsilon})\right)\mathbf{v}$$
$$= \frac{-(1-\lambda)^2}{4}\hat{\delta}_m^{1+2\varepsilon}\left(1+O(\hat{\delta}_m^{\varepsilon})\right)\mathbf{v}, \tag{64}$$

which as $\lambda \neq 1$ gives a sharp estimate in (8) for $\varepsilon \in (0,1]$ (up to the asymptotics of the second component $f_{l2}(\bar{t})$ in (60) - see next step).

c) We demonstrate now that the second expression $f_{l2}(\bar{t})$ in (60) has higher convergence order than $\hat{\delta}_m^{1+2\varepsilon}$. For the latter, it suffices to show that the expression $(1/6)\psi_i[t_i,t_{i+1}] = \hat{\delta}_m^{-1+\lambda}+O(\hat{\delta}_m^{-1+\lambda+\varepsilon})$ (see (61)) has slower asymptotics than the expression D

$$= \psi_i[t_i,t_{i+1},t_{i+2}]\int_{[0,1]^3}(2\eta(\bar{t})-t_i-t_{i+1})u^2u_1dudu_1du_2. \tag{65}$$

Indeed for $\bar{t} = (t_i+t_{i+1})/2$ we have $2\eta(\bar{t})-t_i-t_{i+1}$

$$= 2\{[(\bar{t}u_2+(1-u_2)t_{i+2})u_1+(1-u_1)t_{i+1}]u+(1-u)t_i\}$$
$$-t_i-t_{i+1}$$
$$= 2\{[(\bar{t}u_2+(1-u_2)t_{i+2})u_1+(1-u_1)t_{i+1}]u\}+(t_i-t_{i+1})$$
$$-2ut_i$$
$$= 2[(\bar{t}u_2+(1-u_2)t_{i+2})uu_1]+2u(t_{i+1}-t_i)-2uu_1t_{i+1}$$
$$+(t_i-t_{i+1})$$
$$= 2\bar{t}uu_1u_2-2uu_1u_2t_{i+2}+2uu_1(t_{i+2}-t_{i+1})+2u(t_{i+1}-t_i)$$
$$+(t_i-t_{i+1})$$
$$= 2uu_1u_2(\bar{t}-t_{i+2})+2uu_1(t_{i+2}-t_{i+1})+2u(t_{i+1}-t_i)$$
$$+(t_i-t_{i+1})$$
$$= uu_1u_2((t_i-t_{i+2})+(t_{i+1}-t_{i+2}))+2uu_1(t_{i+2}-t_{i+1})$$
$$+2u(t_{i+1}-t_i)+(t_i-t_{i+1}).$$

Coupling the latter with (55) yields the integral from (65) equal to:

$$\int_{[0,1]^3}(u^3u_1^2u_2((t_i-t_{i+2})+(t_{i+1}-t_{i+2}))$$
$$+2u^3u_1^2(t_{i+2}-t_{i+1})+2u^3u_1(t_{i+1}-t_i)+u^2u_1(t_i-t_{i+1}))\,dudu_1du_2$$
$$= (1/24)((t_i-t_{i+2})+(t_{i+1}-t_{i+2}))+(1/6)(t_{i+2}-t_{i+1})$$
$$+(1/4)(t_{i+1}-t_i)+(1/6)(t_i-t_{i+1})$$
$$= (-1/24)(t_{i+2}-t_i)+(1/8)(t_{i+2}-t_{i+1})$$
$$+(1/12)(t_{i+1}-t_i)$$
$$= \hat{\delta}_m\left((-1/12)+(1/8)(1-\hat{\delta}_m^{\varepsilon})+(1/12)(1+\hat{\delta}_m^{\varepsilon})\right)$$
$$= \hat{\delta}_m\left((1/8)+O(\hat{\delta}_m^{\varepsilon})\right).$$

Combining the above with (62) and (65) leads to:

$$D = \left((1-\lambda)\hat{\delta}_m^{-2+\varepsilon+\lambda}+(1-\lambda)O(\hat{\delta}_m^{-2+3\varepsilon+\lambda})\right)$$
$$\cdot\hat{\delta}_m\left((1/8)+O(\hat{\delta}_m^{\varepsilon})\right)$$
$$= \frac{1-\lambda}{8}\hat{\delta}_m^{-1+\lambda+\varepsilon}+(1-\lambda)O(\hat{\delta}_m^{-1+2\varepsilon+\lambda}), \tag{66}$$

which yields faster convergence rate by ε than the term $\psi_i[t_i,t_{i+1}]$ (we assumed here that $\lambda \neq 1$). Thus (64) and (66) prove sharpness of (8) for $\varepsilon \in (0,1]$.

Note that for $\lambda = 1$ (by (62)) here $f(t) \equiv \mathbf{0}$ since $\psi_i^{(2)}(t) = 0$ (as the quadratic ψ_i is an affine function) and $\gamma_l^{(3)}(t) = \mathbf{0}$). The sharpness of Th. 4 for $\lambda = 1$ is demonstrated in [10] or [11].

A close inspection of the proof of Th. 4 shows that in fact for γ_l and for sampling (55) the cubic component in $\min\{3,1+\varepsilon\}$ for $\varepsilon \geq 1$ does not occur and the asymptototic order $1+2\varepsilon$ prevails for all $\varepsilon > 0$ (as indeed proved above). Such acceleration is also numerically confirmed in Ex. 3.

In order to prove the sharpness of cubic orders in (8) for $\varepsilon > 1$ (and $\lambda \neq 1$) we consider a cubic curve (71) (see Ex. 3 b)) sampled according to (55). Note that as $\gamma_c'''(t) =$

$(0,6) \neq \mathbf{0}$ and as (59) is always a non-vanishing term of order $\hat{\delta}_m^3$ we have e.g. over $I_0 = [t_0, t_2]$ that $f_c(\eta(\bar{t}))$

$$= O(\hat{\delta}_m^3) O(\hat{\gamma}_2(\psi_0(\eta(\bar{t})))) O(\psi_0^{(1)}(\eta(\bar{t})) O(\psi_0^{(2)}(\eta(\bar{t}))) - O(\hat{\delta}_m^3) \quad (67)$$

with $\bar{t} = (t_0 + t_2)/2$. It is sufficient to show that the first component in (67) has order $O(\hat{\delta}_m^{1+2\varepsilon})$. Repeating the calculation from above carried out for γ_l (upon recalling $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$ and the Binomial Th.) yields:

$$\psi_0[t_0, t_1] = \hat{\delta}_m^{-1+\lambda}(1 + \hat{\delta}_m^{\varepsilon})^{-1+\lambda}(1 + O(\hat{\delta}_m^4)),$$

$$\psi_0[t_1, t_2] = \hat{\delta}_m^{-1+\lambda}(1 - \hat{\delta}_m^{\varepsilon})^{-1+\lambda}(1 + O(\hat{\delta}_m^4)),$$

$$\psi_0[t_0, t_1, t_2] = \hat{\delta}_m^{-2+\lambda+\varepsilon}((\lambda - 1) + (\lambda - 1) O(\hat{\delta}_m^{2\varepsilon})) \cdot (1 + O(\hat{\delta}_m^4)),$$

$$\hat{\gamma}_{2,0}[\hat{t}_0, \hat{t}_1] = \hat{\delta}_m^{1-\lambda}(1 + \hat{\delta}_m^{\varepsilon})^{1-\lambda}\mathbf{v_1},$$

$$\hat{\gamma}_{2,0}[\hat{t}_1, \hat{t}_2] = \hat{\delta}_m^{1-\lambda}(1 - \hat{\delta}_m^{\varepsilon})^{1-\lambda}\mathbf{v_2} \quad (68)$$

and

$$\hat{\gamma}_{2,0}[\hat{t}_0, \hat{t}_1, \hat{t}_2] = \frac{\hat{\delta}_m^{1-2\lambda+\varepsilon}(2(\lambda - 1) + O(\hat{\delta}_m^{2\varepsilon}))}{(1 + \hat{\delta}_m^{\varepsilon})^{\lambda}(1 + O(\hat{\delta}_m^4)) + (1 - \hat{\delta}_m^{\varepsilon})^{\lambda}(1 + O(\hat{\delta}_m^4))}\mathbf{v} = \frac{\hat{\delta}_m^{1-2\lambda+\varepsilon}((\lambda - 1) + (\lambda - 1) O(\hat{\delta}_m^{2\varepsilon}))}{1 + O(\hat{\delta}_m^{\min\{4,2\varepsilon\}})}\mathbf{v}$$
$$= \hat{\delta}_m^{1-2\lambda+\varepsilon}\left((\lambda - 1) + (\lambda - 1) O(\hat{\delta}_m^{2\varepsilon})\right) \cdot \left(1 + O(\hat{\delta}_m^{\min\{4,2\varepsilon\}})\right)\mathbf{v}, \quad (69)$$

where vectors $\mathbf{v_i} = (1, O(1))$ (for $i = 1,2$) and $\mathbf{v} = (1, O(1))$. An analogous analysis as for curve γ_l applied to (68) and (69) renders for the first component in (67) the asymptotics of order $O(\hat{\delta}_m^{1+2\varepsilon})$ (and thus also of order $O(\delta_m^{1+2\varepsilon})$). □

3 Experiments

The tests are conducted in *Mathematica 9.0* (see e.g. [13]) on a 2.4 GHz Intel Core 2 Duo computer with 8 GB RAM. Since $1 = \sum_{i=1}^{m}(t_{i+1} - t_i) \leq m\delta_m$ the following holds $m^{-\alpha} = O(\delta_m^{\alpha})$, for $\alpha > 0$. Hence, the verification of the asymptotics expressed in terms of $O(\delta_m^{\alpha})$ can be performed by examining the claim of Th. 4 in terms of $O(1/m^{\alpha})$ asymptotics.

For a parametric regular curve $\gamma : [0,1] \to E^n$ $\lambda \in [0,1]$ and m varying between $m_{min} \leq m \leq m_{max}$ the i-th component of the error for γ estimation is defined here according to:

$$E_m^i = \sup_{t \in [t_i, t_{i+2}]} \|(\hat{\gamma}_{2,i} \circ \psi_i)(t) - \gamma(t)\| = max_{t \in [t_i, t_{i+2}]} \|(\hat{\gamma}_{2,i} \circ \psi_i)(t) - \gamma(t)\|,$$

as $\tilde{E}_m^i(t) = \|(\check{\gamma}_{2,i} \circ \psi_i)(t) - \gamma(t)\| \geq 0$ is continuous over each sub-interval $[t_i, t_{i+2}] \subset [0,1]$. The maximal value E_m of $\tilde{E}_m(t)$ (the track-sum of $\tilde{E}_m^i(t)$), for each $m = 2k$ (here $k = 1,2,3,\ldots,m/2$) is found by using *Mathematica* optimization built-in functions: *Maximize* or *FindMinimum* (the latter applied to $-\tilde{E}_m(t)$). From the set of *absolute errors* $\{E_m\}_{m=m_{min}}^{m_{max}}$ the numerical estimate $\bar{\alpha}(\lambda)$ of genuine order $\alpha(\lambda)$ is subsequently computed by using *a linear regression* to the pair of points $(\log(m), -\log(E_m))$ (see also [3]). Since piecewisely $deg(\hat{\gamma}_2) = 2$ the number of interpolation points $\{q_i\}_{i=0}^{m}$ is odd i.e. $m = 2k$ as indexing runs over $0 \leq i \leq m$. The *Mathematica* built-in functions *LinearModelFit* renders the coefficient $\bar{\alpha}(\lambda)$ from the computed regression line $y(x) = \bar{\alpha}(\lambda)x + b$ based on pairs of points $\{(\log(m), -\log(E_m))\}_{m=m_{min}}^{m_{max}}$. Note that as indicated in [11] the tested regular curves need not be parameterized exclusively by arc-length. Namely, given our interpolation scheme both regular curve γ and its reparameterized version by arc-length $\gamma \circ \theta$ (see also [14]) yields the same asymptotics for trajectory estimation (which in particular applies to Th. 4). Finally, recall that as justified in Th. 4 any ε-uniform sampling renders asymptotically ψ_i as reparameterization of $[t_i, t_{i+2}]$ into $[\hat{t}_i, \hat{t}_{i+2}]$ - recall that by Remark 5 the tests can equally use normalized or unnormalized exponential parameterizations (6).

In the next steps we test experimentally the asymptotics established in Th. 4 together with the sharpness established by Ex. 2. First we verify the latter.

Example 3. a) Consider a regular straight line (parameterized by arc-length):

$$\gamma_l(t) = \left(\frac{t}{\sqrt{5}}, \frac{2t}{\sqrt{5}}\right) \subset E^2 \quad (70)$$

for $t \in [0,1]$, sampled according to (55), where $t_0 = 0$ and $t_m = 1$ - see Fig. 1 for the distribution of $\{\gamma_l(t_i)\}_{i=0}^{m}$ with $\varepsilon = 0.5$ and $m = 12$.

Recall that case $\lambda = 1$ is excluded in Ex. 2. The quadratic ψ_i is a genuine reparameterization (see Step 1). The linear regression is applied to $m_{min} = 101 \leq m \leq m_{max} = 121$ and the results for computed $\bar{\alpha}_{\varepsilon}(\lambda) \approx \alpha_{\varepsilon}(\lambda) = \min\{3, 1 + 2\varepsilon\}$ are presented in Tab. 1. Note that *sharpness or nearly sharpness* of Th. 4 is confirmed herein for $\varepsilon \in (0,1]$ as proved in Ex. 2. In fact as indicated also in Ex. 2 the sharp result for γ_l and samplings (55) should coincide with $1 + 2\varepsilon$ for all $\varepsilon > 0$. Indeed the latter is supported by the numerical estimates $\bar{\alpha}_{\varepsilon}(\lambda)$ listed in Tab. 2.

b) Consider *a cubic curve* $\gamma_c : [0,1] \to E^2$ defined as:

$$\gamma_c(t) = (t, t^3) \quad (71)$$

sampled according to (55). Visibly γ_c is a regular curve. The numerical cubic estimates for $\varepsilon \geq 1$ conducted for

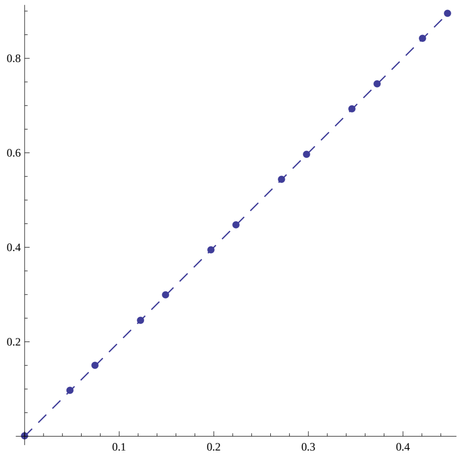

Fig. 1: The plot of the straight line γ_l from (70) sampled according to (55), for $m = 12$ and $\varepsilon = 0.5$.

Table 1: Computed $\bar{\alpha}_\varepsilon(\lambda) \approx \alpha_\varepsilon(\lambda) = \min\{3, 1+2\varepsilon\}$ for γ_l from (70) sampled along (55) and interpolated by $\hat{\gamma}_2$ with some discrete values $\lambda \in [0,1)$ and $\varepsilon \in (0,1]$.

λ	$\varepsilon = 0.1$	$\varepsilon = 0.33$	$\varepsilon = 0.5$	$\varepsilon = 0.7$	$\varepsilon = 0.9$	$\varepsilon = 1.0$
0.00	1.47	1.80	2.10	2.46	2.85	3.04
0.10	1.45	1.80	2.10	2.46	2.85	3.04
0.33	1.42	1.80	2.10	2.46	2.85	3.04
0.50	1.39	1.80	2.10	2.46	2.85	3.04
0.70	1.37	1.79	2.10	2.47	2.85	3.04
0.90	1.36	1.79	2.10	2.47	2.85	3.04
$\alpha_\varepsilon(\lambda)$	**1.20**	**1.66**	**2.00**	**2.40**	**2.80**	**3.00**

Table 2: Computed $\bar{\alpha}_\varepsilon(\lambda) \approx \alpha_\varepsilon(\lambda) = 1+2\varepsilon$ for γ_l from (70) sampled along (55) and interpolated by $\hat{\gamma}_2$ with some discrete values $\lambda \in [0,1)$ and $\varepsilon \geq 1$.

λ	$\varepsilon = 1.0$	$\varepsilon = 1.5$	$\varepsilon = 1.7$	$\varepsilon = 2.0$	$\varepsilon = 2.5$	$\varepsilon = 2.7$
0.33	3.04	4.04	4.44	5.05	6.04	6.37
0.50	3.04	4.04	4.44	5.05	6.03	6.30
$\alpha_\varepsilon(\lambda)$	**3.00**	**4.00**	**4.40**	**5.00**	**6.00**	**6.40**

Table 3: Computed $\bar{\alpha}_\varepsilon(\lambda) \approx \alpha_\varepsilon(\lambda) = 3$ for γ_c from (71) sampled along (55) and interpolated by $\hat{\gamma}_2$ with some discrete values $\lambda \in [0,1)$ and $\varepsilon \geq 1$.

λ	$\varepsilon = 1.0$	$\varepsilon = 1.5$	$\varepsilon = 2.0$	$\varepsilon = 3.0$	$\varepsilon = 4.0$	$\varepsilon = 5.0$
0.33	3.04	3.03	3.02	3.03	3.03	3.04
0.50	3.02	3.04	3.03	3.03	3.03	3.04
$\alpha_\varepsilon(\lambda)$	**3.00**	**3.00**	**3.00**	**3.00**	**3.00**	**3.00**

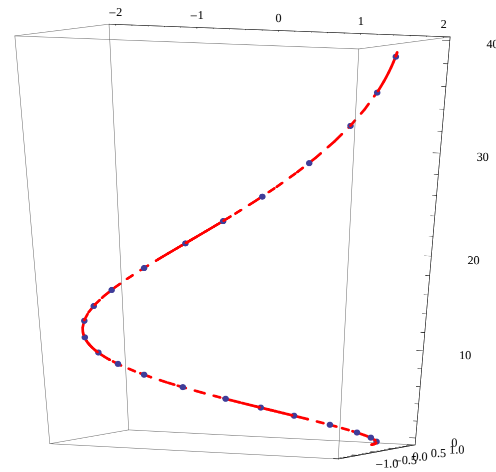

Fig. 2: The plot of the helix γ_h from (72) sampled according to (73), for $m = 22$ and $\varepsilon = 0.5$.

$100 \leq m \leq 121$ shown in Tab. 3 confirm the sharpness of Th. 4. $\square$

The next example refers to the regular spatial curve in E^3.

Example 4. We verify now the sharpness of Th. 4 for *a quadratic elliptical helix* $\gamma_h : [0,1] \to E^3$:

$$\gamma_h(t) = (2\cos(2\pi t), \sin(2\pi t), 4\pi^2 t^2), \quad (72)$$

sampled ε-uniformly (5) (with $\phi = id$) according to:

$$t_i = \begin{cases} \frac{i}{m}, & \text{if } i \text{ even;} \\ \frac{i}{m} + \frac{1}{2m^{1+\varepsilon}}, & \text{if } i = 4k+1; \\ \frac{i}{m} - \frac{1}{2m^{1+\varepsilon}}, & \text{if } i = 4k+3. \end{cases} \quad (73)$$

Fig. 2 illustrates the curve γ_h sampled along (73) for $\varepsilon = 0.5$ and $m = 22$. Recall again that, by Th. 4 the function ψ_i is a reparameterization. All tests conducted in this example resort to the linear regression applied for $m_{min} = 101 \leq m \leq m_{max} = 121$. The corresponding computed estimates $\bar{\alpha}_\varepsilon(\lambda) \approx \alpha_\varepsilon(\lambda) = \min\{3, 1+2\varepsilon\}$ are presented in Tab. 4.

Again all obtained results are consistent with the asymptotics established in Th. 4. The sharpness of (8) is also generically confirmed. $\square$

Some combinations of curves $\gamma \in C^4([0,1])$ and ε-uniform samplings (5) may provide an extra acceleration in asymptotics in comparison with those from Th. 4. Such potential situation is shown in the next example.

Example 5. Consider *a planar regular convex spiral* $\gamma_{sp} : [0,1] \to E^2$ defined as:

$$\gamma_{sp}(t) = ((6\pi - 5\pi t)\cos(5\pi t), (6\pi - 5\pi t)\sin(5\pi t)) \quad (74)$$

Table 4: Estimated $\bar{\alpha}_\varepsilon(\lambda) \approx \alpha_\varepsilon(\lambda) = \min\{3, 1+2\varepsilon\}$ (with $\lambda \in [0,1)$) and $\bar{\alpha}_\varepsilon(1) \approx \alpha_\varepsilon(1) = 3$ for γ_h from (72) sampled along (73) and interpolated by $\hat{\gamma}_2$ for some discrete values $\lambda \in [0,1]$ and $\varepsilon \in (0,1]$.

λ	$\varepsilon=0.1$	$\varepsilon=0.33$	$\varepsilon=0.5$	$\varepsilon=0.7$	$\varepsilon=0.9$	$\varepsilon=1.0$
0.00	1.26	1.74	2.10	2.54	2.96	3.01
0.10	1.26	1.74	2.09	2.54	2.97	3.01
0.33	1.24	1.72	2.07	2.93	2.93	2.95
0.50	1.23	1.70	2.06	3.01	3.01	3.04
0.70	1.20	1.64	2.94	2.94	2.94	3.19
0.90	1.15	2.89	2.89	2.89	2.89	3.22
$\alpha_\varepsilon(\lambda)$	**1.20**	**1.66**	**2.00**	**2.40**	**2.80**	**3.00**
1.00	2.89	2.91	2.92	2.93	2.88	3.21
$\alpha_\varepsilon(1)$	**3.00**	**3.00**	**3.00**	**3.00**	**3.00**	**3.00**

Table 5: Estimated $\bar{\alpha}_\varepsilon(\lambda) \approx \alpha_\varepsilon(\lambda) = \min\{3, 1+2\varepsilon\}$ (with $\lambda \in [0,1)$) and $\bar{\alpha}_\varepsilon(1) \approx \alpha_\varepsilon(1) = 3$ for γ_{sp} from (74) sampled along (73) and interpolated by $\hat{\gamma}_2$ for some discrete values $\lambda \in [0,1]$ and $\varepsilon \in (0,1]$.

λ	$\varepsilon=0.1$	$\varepsilon=0.33$	$\varepsilon=0.5$	$\varepsilon=0.7$	$\varepsilon=0.9$	$\varepsilon=1.0$
0.00	1.25	2.07	2.80	2.96	2.97	2.97
0.10	1.26	2.16	2.84	2.96	2.97	2.97
0.33	1.33	2.44	2.91	2.97	2.97	2.98
0.50	1.45	2.67	2.95	3.97	2.97	2.98
0.70	1.87	2.89	2.97	2.97	2.97	2.98
0.90	2.82	2.99	2.99	2.98	2.97	2.98
$\alpha_\varepsilon(\lambda)$	**1.20**	**1.66**	**2.00**	**2.40**	**2.80**	**3.00**
1.00	2.99	3.01	2.99	2.98	2.96	2.97
$\alpha_\varepsilon(1)$	**3.00**	**3.00**	**3.00**	**3.00**	**3.00**	**3.00**

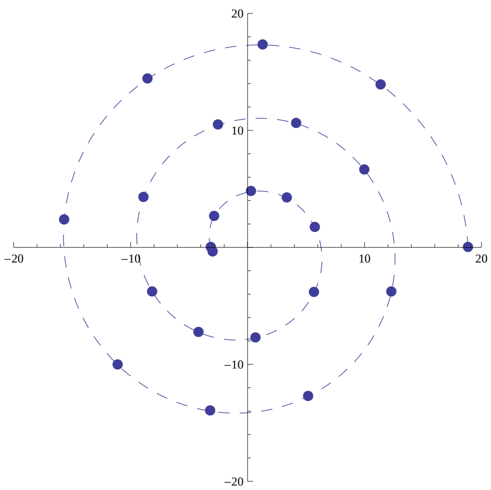

Fig. 3: The plot of the spiral γ_{sp} from (74) sampled according to (73), for $m = 22$ and $\varepsilon = 0.33$.

sampled in accordance to (73). Fig. 3 illustrates γ_{sp} coupled with (73) for $\varepsilon = 0.33$ and $m = 22$. The verification for sampling (73) enforcing ψ_i to be a reparameterization (proved earlier to be automatically fulfilled) can be accomplished as in the previous example (see also (19)). For the numerical assessment of $\alpha_\varepsilon(\lambda)$, as previously a linear regression is applied to $101 \leq m \leq 121$. The relevant numerical results are listed in Tab. 5.

Evidently most of the experiments from Tab. 5 indicate faster convergence rates as opposed to those established in Th. 4. □

4 Conclusion

In this paper we extend the existing results for trajectory estimation via *piecewise-quadratic interpolation based on reduced data sampled ε-uniformly*. Our analysis focuses on *the exponential parameterization* (6) which depends on a parameter $\lambda \in [0,1]$. Exponential parameterization is commonly used in computer graphics for curve modeling - see e.g. [4]. The case when $\lambda = 0$ is discussed in [9]. The opposite one with $\lambda = 1$, refers to the cumulative chords and general admissible samplings (1) which is already analyzed e.g. in [3] or [10]. A recent result [11] (established for samplings (3) and curves $\gamma \in C^3([0,T])$) addresses the remaining cases of $\lambda \in (0,1)$ by proving that there is no acceleration in trajectory estimation, and that the respective convergence orders $\alpha(\lambda) = 1$, for all $\lambda \in [0,1)$ have a discontinuity at $\lambda = 1$ with a jump to $\alpha(1) = 3$.

However, a further acceleration can be achieved for ε-uniform samplings (5) and $\lambda = 0$ (see [9]), with *sharp* orders $\alpha_\varepsilon(0) = \min\{3, 1+2\varepsilon\}$ claimed for trajectory estimation (with $\varepsilon > 0$). *The main result* of this paper (i.e. Th. 4 and Ex. 2) extends the latter to all $\lambda \in [0,1)$ combined with ε-uniform samplings. As demonstrated the accelerated convergence orders $\alpha_\varepsilon(\lambda) = \min\{3, 1+2\varepsilon\}$ are not dependent on $\lambda \in [0,1)$ but merely on ε. Again for $\lambda \in [0,1)$ with $0 < \varepsilon < 1$ at $\lambda = 1$ we have a discontinuous jump in convergence order from $\alpha_\varepsilon(\lambda) = 1 + 2\varepsilon$ to $\alpha_\varepsilon(1) = 3$. Such discontinuity is removed once $\varepsilon \geq 1$ as then cubic orders hold for both $\lambda = 1$ and $\lambda \in [0,1)$. This paper proves also that a natural candidate for reparameterization of $[t_i, t_{i+2}]$ into $[\hat{t}_i, \hat{t}_{i+2}]$ i.e. a Lagrange quadratic ψ_i satisfying $\psi_i(t_{i+j}) = \hat{t}_{i+j}$ with $j = 0,1$ (see (6)) forms a genuine reparameterization for all ε-uniform samplings. On the other hand, the latter does not always hold for arbitrary more-or-less uniform samplings (3) as shown in [11]. It should be mentioned that Th. 4 extends also to the case when $\varepsilon = 0$ (with (8) still sharp), upon imposing extra constraints on samplings (we omit the analysis). The ε-uniformly sampled reduced data Q_m in the context of the asymptotics of length estimation for an arbitrary regular curve in E^n has been recently discussed in [15].

A possible extension of this work is to invoke smooth interpolation schemes (see [6]) combined with reduced data exponential parameterization (see [4]). Certain clues may be given in [16], where complete C^2 splines are dealt with for $\lambda = 1$, to obtain the fourth orders of convergence in length estimation. The analysis of C^1 interpolation for reduced data with cumulative chords (i.e. again with $\lambda = 1$) can additionally be found in [3] or [17].

There are also other parameterizations applied predominantly on sparse data (applicable also on dense Q_m) - see e.g. the so-called *blending parameterization* [18] or *monotonicity or convexity preserving ones* [4]. The alternative approach is discussed in [19].

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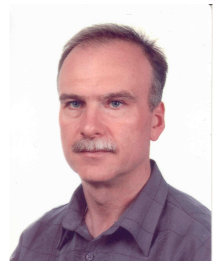

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