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Consistency of Estimators in Partially Observed Subcritical Branching Processes with Immigration

I. Rahimov[*]

Department of Mathematics and Statistics, Zayed University, P.O.Box 19282, Dubai, UAE.

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Abstract: It is known that in subcritical branching process with stationary immigration the average population size for the first n generations and the ratio of the reproduction process to the total progeny are strongly consistent estimators for the mean of the stationary distribution and for the offspring mean, respectively. We prove that the same estimators remain strongly consistent, if we have only partial observations of the population and the number of immigrants. We also show that the rates of convergence of the estimators to the true values of the parameters are the same as in the case of complete observation.

Keywords: branching process, subcritical, restricted observation, offspring mean, stationary distribution, random sum, limit theorems.

1 Introduction

We consider a discrete time branching stochastic process $W_n, n \geq 0, W_0 = 1$, defined by two families of independent, nonnegative integer valued random variables $\{X_{ni}, (n,i) \in \mathcal{N}^2\}, \mathcal{N} = \{1, 2...\}$ and $\{\nu_k, k \geq 1\}$ recursively as

$$W_{n+1} = \sum_{i=1}^{W_n} X_{ni} + \nu_{n+1}, \quad n \geq 0. \tag{1.1}$$

Assume that X_{ni} have a common distribution for all n and i, and families $\{X_{ni}\}$ and $\{\nu_n\}$ are independent. We also assume that $\{\nu_k, k \geq 1\}$ are independent and identically distributed. Variables X_{ni} will be interpreted as the number of offspring of the ith individual in nth generation and ν_n is the number of immigrating individuals in the nth generation. Then, W_n can be considered as the size of nth generation of the population.

Suppose $m = EX_{ni}$ is the mean number of offspring of a single individual. Process W_n is called *subcritical*, *critical* or *supercritical* depending on $m < 1, m = 1$ or $m > 1$, respectively. The independence assumption for the families $\{X_{ni}\}$ and $\{\nu_n\}$ means that reproduction and immigration processes are independent.

Since the behavior of the branching process is mostly determined by the offspring mean m, the problem of estimation of this parameter is important. Therefore estimation of the offspring and immigration parameters in the branching process with immigration have been an active area of the research for a long time. As a result of this activity, it has been established that a maximum likelihood approach leads to useful results, if the number of immigrating individuals $\{\nu_k, k \geq 1\}$ and all offspring sizes $\{X_{ni}, n, i \geq 1\}$ are observable. Later, it turned out that the offspring mean can successfully be estimated, if population sizes $\{W_i, 1 \leq i \leq n\}$ up to some generation are observed. However, in applications very often not all individuals existing in the population will be observed. As a result, possibility of estimating the offspring mean based on partial observations is of interest.

In estimating the offspring mean in the process with immigration based on full observation the analogy between the immigration-branching process and the first order autoregressive process is often useful (see [3,4]) However, there is no such a similarity, if we have only partial observations of the population sizes, which is due to complexity of the dependence in partially observed process.

The partially observed branching process is defined as following. Each of W_n individuals existing at time n independently from others can be detected with probability $\theta, 0 < \theta < 1$ and remains undetected with probability $1 - \theta$.

[*] Corresponding author e-mail: Ibrahim.Rahimov@zu.ac.ae

Assume that the offspring distributions of both detected and undetected individuals may change after inspection. Let $\{\xi_{ni},(n,i) \in \mathcal{N}^2\}$ be a family of i.i.d. Bernoulli random variables with a probability of success θ and $\{X_{ni}^{(j)},(n,i) \in \mathcal{N}^2\}, j = 1,2$ be two independent families of i.i.d. random variables taking nonnegative integer values and these families may follow different probability distributions for $j = 1$ and 2. Assume also that families $\{\xi_{ni},(n,i) \in \mathcal{N}^2\}$ and $\{X_{ni}^{(j)},(n,i) \in \mathcal{N}^2\}$ are independent for all values of n,i and j. If we take

$$X_{ni} = X_{ni}^{(1)}(1-\xi_{ni}) + X_{ni}^{(2)}\xi_{ni}$$

in (1.1), we obtain new process

$$Z_0 = 1,$$

$$Z_{n+1} = \sum_{i=1}^{Z_n} X_{ni}^{(1)}(1-\xi_{ni}) + \sum_{i=1}^{Z_n} X_{ni}^{(2)}\xi_{ni} + \nu_{n+1} \quad n \geq 0. \tag{1.2}$$

We note that $\{Z_n, n \geq 0\}$ is the standard branching process with immigration and with a special offspring distribution. The partially observed branching process with immigration is defined as

$$Y_{n+1} = \sum_{i=1}^{Z_n} \xi_{ni},\ n \geq 0.$$

The partially observed (or restricted) branching processes have been considered by many authors mostly in the framework of applications in epidemic modeling. It is known that ”susceptible-infectious-removed” (SIR) epidemic model can successfully be approximated by branching processes, when the initial number of susceptible individuals is large. In [1] a systematic study of SIR models, based on the branching approximation, is provided.

Problems related to statistical inference based on partially observed branching processes have been considered in [5, 6,7] and [10]. In particular, in [6] it has been demonstrated that, if one has a binomially distributed subset of observations of each generation, one will be able to estimate three functions of parameters of the offspring distribution. We also note that these papers concentrate on possibility of using traditional Lotka-Nagaev and Harris type ratio estimators to estimate the offspring mean when the observations are restricted.

In this note we consider the problem of estimation of parameters of the process with immigration based on partial observations of the branching process. We concentrate on subcritical case. It is well known that when $m < 1$ the process W_n defined in (1.1) has a unique stationary probability distribution, if and only if $E(log^+\nu_n)$ is finite. If in addition $E(\nu_n) < \infty$, then the stationary distribution has a finite mean $\mu = E(\nu_n)/(1-m)$. A. Pakes [9] has proposed an strongly consistent and asymptotically normal estimator for μ using his results about the total progeny of the process. Later it was shown in [8] that if there is an observation of population sizes $\{W_i, i = 1,2,...n\}$ and the numbers of immigrating individuals $\{\nu_i, i = 1,2,...n\}$ one can construct the maximum likelihood estimator for the offspring mean m, which is asymptotically normal when $n \to \infty$. The question, which we address in this paper is simple: will the estimators proposed in [8] and [9] remain consistent, if we have only partially observed population sizes and the numbers of immigrating individuals Another question of interests in this note is the rate of convergence of the estimators to the true value of the parameters and asymptotic normality of the estimators.

We prove that the estimators proposed by Pakes and Nanthi based on partial observations remain strongly consistent. We also obtain results which give the rate of convergence of these estimators. What concerns the asymptotic normality of the estimators, our results show that the difference between estimators and the true values of parameters normed by $n^{1/2}$ converges in distribution to a sum of several normal random variables which are, generally speaking, not necessarily independent. The same kind of results are obtained in [6] for the partially observed processes without immigration, which is because of the more complexity of the dependence in the partially observed process. It is clear that one has to consider some modified estimators in this situation. Here the ”skipping” method exploited (first used in [6]) in [5] may be useful. Unlike the proofs in [5] and [6], our proofs use a central limit theorem (CLT) for a random sums of dependent random variables (see [11], p.13), and do not relay Scott’s CLT [12].

In Section 2 we introduce the estimators and formulate the main results. In Section 3 we provide the CLT from [11] in a appropriate form for a ready reference. Certain preliminary results and proofs of the main theorems are given in Section 4.

2 Main results

It is not difficult to see that the new process Z_n is a branching process with the same immigration probability generating function as (1.1) and with offspring generating function

$$f(s) = (1-\theta)f_1(s) + \theta f_2(s), \tag{2.1}$$

where $f_1(s) = Es^{X_{11}^{(1)}}$ and $f_2(s) = Es^{X_{11}^{(2)}}$ are offspring generating functions of branching processes started by detected and undetected individuals, respectively. From here we easily find that the offspring mean $m =: EX_{ni}$ and the variance $\sigma^2 =: VarX_{ni}$ are defined as follows:

$$m = (1-\theta)\mu_1 + \theta\mu_2,$$

$$\sigma^2 = (1-\theta)\sigma_1^2 + \theta\sigma_2^2 + (1-\theta)\theta(\mu_1-\mu_2)^2,$$

where $\mu_i = EX_{11}^{(i)}$ and $\sigma_i^2 = VarX_{11}^{(i)}$ for $i = 1,2$. Throughout the paper we assume that the immigration mean $\lambda := E\nu_i$ and the variance $\gamma^2 := Var\nu_i$ are finite.

We also denote $p_j^{(i)} = P\{X_{11}^{(i)} = j\},\ j = 0,1,...,\ i = 1,2$. To exclude the trivial situations, we assume throughout the paper that

$$p_0^{(i)} + p_1^{(i)} < 1,\ p_j^{(i)} \neq 1, j = 0,1,...$$

for $i = 1,2$. Throughout in the paper $N(a,b^2)$ denotes a normal random variable with mean a and variance b^2.

As it was already mentioned, for the completely observed process defined by (1.1) A. Pakes [9] has shown that

$$\hat{\mu}_n = \frac{1}{n}\sum_{i=0}^{n} W_i$$

is strongly consistent estimator for the mean of the stationary distribution $\mu = \lambda(1-m)^{-1}$. Under the conditions $m < 1$ and both σ^2 and γ^2 are finite, the same author has established that $\sqrt{n}(\hat{\mu}_n - \mu)$ as $n \to \infty$ converges in distribution to a normal random variable with mean zero and the variance $(\mu\sigma^2 + \gamma^2)/(1-m)^2$.

What concerns the offspring mean, in the case when the population sizes $\{W_i, i = 1,2,...n\}$ and the numbers of immigrating individuals $\{\nu_i, i = 1,2,...n\}$ are completely observable, it was shown in [8] (see also [13]) that

$$\hat{m}_n = \frac{\sum_{i=1}^{n}(W_i - \nu_i)}{\sum_{i=1}^{n} W_{i-1}} \tag{2.2}$$

is a maximum likelihood estimator for m and $\sqrt{n}(\hat{m}_n - m)$ as $n \to \infty$ converges in distribution to a normal random variable with mean zero and the variance σ^2/μ.

To estimate m based on a partial observation, we define by η_n the number of observed individuals immigrating in the $(n-1)st$ generation. Let $\{\zeta_{ni}, (n,i) \in \mathscr{N}^2\}$ be a family of i.i.d. Bernoulli random variables with the same probability of success θ and independent of $\{X_{ni}^{(j)}, (n,i) \in \mathscr{N}^2\}, j = 1,2$ and $\{\xi_{ni}, (n,i) \in \mathscr{N}^2\}$. Then, it is natural to define η_n as

$$\eta_{n+1} = \sum_{j=1}^{\nu_n} \zeta_{nj},\ n \geq 1.$$

We consider the following estimators for μ and m

$$\hat{t}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i,\ \ \hat{a}_n = \frac{\sum_{i=2}^{n+1}(Y_i - \eta_i)}{\sum_{i=1}^{n} Y_i} \tag{2.3}$$

based on the partial observations of the reproduction and the immigration processes.

First we provide result related to the estimator of the mean of the stationary distribution of the process.

Theorem 1. *Let $m < 1$ and $\lambda < \infty$.*
a) Then $\hat{t}_n$ is a strongly consistent estimator for $\theta\mu$ i.e. $\hat{t}_n \to \theta\mu$ almost surely as $n \to \infty$, where μ is the mean of the stationary distribution of Z_n.
b) If in addition $\sigma^2 < \infty$ and $\gamma^2 < \infty$, then $\sqrt{n}(\hat{t}_n - \theta\mu)$ can be represented as a sum of two asymptotically normal as

$n \to \infty$ random variables with zero means and finite variances $\theta(\mu\sigma^2+\gamma^2)/(1-m)^2$ and $\mu\theta(1-\theta)$.

Next result is related to estimator of the offspring mean m. To state it we denote $\Delta^2 = \lambda\theta(1-\theta)+(\gamma\theta)^2$.

Theorem 2. *Let $m<1$ and $\lambda<\infty$.*
a) Then $\hat{a}_n$ is a strongly consistent estimator for m.
b) If in addition $\sigma^2<\infty$ and $\gamma^2<\infty$, then $\sqrt{n}(\hat{a}_n-m)$ can be represented as a sum of four asymptotically normal as $n\to\infty$ random variables with zero means and finite variances $(1-\theta)/\mu\theta$, $m^2(1-\theta)/\mu\theta$, σ^2/μ and $(\Delta/\mu\theta)^2$.

3 Random sum of dependent variables

We now provide a central limit theorem for multiple sums of the random number of dependent random variables, which was proved in [11] (see page 13). The notations in this section are independent from other sections of the paper and will be used to formulate the theorem of this section only. Let $\{\xi_{ij}(n),(i,j)\in\mathscr{N}^2\}$ be a family of (generally speaking dependent) random variables for each $n\in\mathscr{N}_0=\{0,1,2,..\}$ and $\{\nu_i(n), i\in\mathscr{N}\}$ be certain family of integer valued random variables taking values from $\mathscr{N}_0$ and defined on a probability space $\{\Omega,F,P\}$. We consider the following sum

$$S_n=\sum_{i=1}^{n}\sum_{j=1}^{\nu_i(n)}\xi_{ij}(n).$$

Let $\{\mathscr{A}_{ij}(n),(i,j)\in N^2\}$ be a family of σ-algebras, such that variable $\xi_{ij}(n)$ is measurable with respect to $\mathscr{A}_{ij}(n)$ for each pair $(i,j)\in N^2$ and

$$F_{kl}(n)=\prod_{i=1}^{k-1}\prod_{j=1}^{\nu_i(n)}\mathscr{A}_{ij}(n)\times\prod_{j=1}^{l}\mathscr{A}_{kj}(n)\times\mathscr{A}_0,$$

where $\mathscr{A}_0$ is a certain σ-algebra such that $\mathscr{A}_0\subset\mathscr{A}_{11}(n)$ for any n and direct products of a random number of σ-algebras, we understand as usual:

$$\prod_{j=1}^{\nu}\mathscr{A}_j(n)=\{A\in F: A\cap\{\nu=l\}\}\in\prod_{j=1}^{l}\mathscr{A}_j(n).$$

We assume that for any l the random variables $\nu_k(n)$ such that $\{\nu_k(n)\le l\}\in F_{kl}(n)$, which means that it is measurable with respect to σ-algebra generated by variables $\{\xi_{ij}(n), i\le k-1, j\le\nu_i(n)\}$ or a stopping time with respect to $\{\xi_{kj}(n), j\ge 1\}$.

We denote $E_{kl}[\xi]=E[\xi|F_{kl}(n)]$ the conditional mean with respect to $F_{kl}(n)$ and

$$\theta_{ij}(n)=E_{ij-1}[\xi_{ij}(n)],\ Y_{ij}(n)=\xi_{ij}(n)-\theta_{ij}(n), \tag{3.1}$$

$$\sigma_{ij}^2(n)=E_{ij-1}[Y_{ij}^2(n)],\ g(\xi,x)=\xi^2\chi(\xi\le x), \tag{3.2}$$

$$L(t,x)=\begin{cases}(e^{itx}-1-itx)x^{-2}, & \text{if } x\neq 0,\\ -\frac{t^2}{2}, & \text{if } x=0.\end{cases}$$

$$\pi(t)=\exp\{it\gamma+\int_{-\infty}^{\infty}L(t,x)dK(x)\}, \tag{3.3}$$

where γ is some $\mathscr{A}_0$-measurable random variable, $\chi(A)$ denotes the indicator function of event A and $K(x)=K(x,\omega)$ is a function mapping $R\times\Omega$ into $R=(-\infty,\infty)$ such that, it is an $\mathscr{A}_0$-measurable random variable for each x and a bounded non-decreasing function of x for almost all fixed ω and $K(\infty,\omega)=0$.

Theorem A. *Let $\{\nu_k(n)\le l\}\in F_{kl}(n)$ for any l and there is a finite $\mathscr{A}_0$-measurable random variable T such that for some sequence of positive numbers $\{B_n, n\ge 1\}$ as $n\to\infty$*

$$P\left\{\frac{1}{B_n^2}\sum_{i=1}^{n}\sum_{j=1}^{\nu_i(n)}\sigma_{ij}^2(n)>T\right\}\to 0, \tag{3.4}$$

$$\frac{1}{B_n^2}\max_{1\le i\le n}\max_{1\le j\le \nu_i(n)}\sigma_{ij}^2(n)\xrightarrow{P}0, \tag{3.5}$$

$$\frac{1}{B_n}\sum_{i=1}^{n}\sum_{j=1}^{\nu_i(n)}\theta_{ij}(n)\xrightarrow{P}\gamma, \tag{3.6}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{\nu_i(n)}E_{ij-1}\left[g(B_n^{-1}Y_{ij}(n),x)\right]\xrightarrow{P}K(x). \tag{3.7}$$

Then as $n\to\infty$

$$\frac{S_n}{B_n}\xrightarrow{d}W, \tag{3.8}$$

where the characteristic function of W is $E(\pi(t))$.

We note that in [11] Theorem A is proved for more general scheme of summation generated by $\xi_{i_1 i_2 \dots i_r}(n),(i_1,i_2,\dots,i_r)\in\mathcal{N}^r$, which are the same type as $\xi_{ij}(n)$. Here we provide the theorem in the form, which is appropriate for our proofs.

4 Proofs of the theorems

We start with a simple but important result, which was first proved in [6] for the process without immigration.

Proposition 1. *If $m<1$ and $\lambda\in(0,\infty)$, then*

$$\frac{\sum_{i=1}^{n}Y_i}{\sum_{i=1}^{n}Z_{i-1}}\to\theta \tag{4.1}$$

almost surely (a.s.) as $n\to\infty$.

Remark. Note that in [6] the authors has shown (4.1) on the explosion set $\{lim_{n\to\infty}X_n=\infty\}$, where X_n is the process without immigration. Our proofs here show that their result remains true on the set $\{lim_{n\to\infty}\sum_{i=1}^{n}X_i=\infty\}$ as well.

Proof. We use the traditional approach, based on a strong law of large numbers from [2] (see Theorem 2.18, p.35). Below we provide it in a appropriate form for ready reference.

Theorem B. *Let $\{T_n=\sum_{i=1}^{n}X_i,\ \mathfrak{F}_n,n\ge1\}$ be a martingale and $\{U_n,n\ge1\}$ a nondecreasing sequence of positive random variables such that U_n is $\mathfrak{F}_{n-1}$-measurable for each n. If $1\le p\le2$, then $lim_{n\to\infty}T_n/U_n=0$ a.s. on the set*

$$\{lim_{n\to\infty}U_n=\infty,\ \sum_{i=1}^{\infty}U_i^{-p}E[|X_i|^p|\mathfrak{F}_{i-1}]<\infty\}.$$

It follows from Pakes results [9] that under our conditions

$$\sum_{i=1}^{n}Z_{i-1}\to\infty$$

a.s. as $n\to\infty$. Therefore we can apply Theorem B to the sum $T_n=\sum_{i=1}^{n}[Y_i-\theta Z_{i-1}]$. Let $\mathfrak{F}_n$ be the σ-algebra generated by $\{Z_0,Z_1,\dots,Z_n,Y_1,Y_2,\dots,Y_n\}$ for each $n\ge1$. Then we have

$$E[T_n|\mathfrak{F}_{n-1}]=T_{n-1}+E[Y_n-\theta Z_{n-1}|\mathfrak{F}_{n-1}].$$

It follows from the definition of Y_n that the second term on the right side of last equality equals zero. On the other hand, one can easily show that $ET_n^2=\theta(1-\theta)\sum_{i=0}^{n-1}EZ_i$. Therefore, under our assumptions $ET_n^2<\infty$ for each $n\ge1$, i.e.$\{T_n,\mathfrak{F}_n,\ n\ge1\}$ is a martingale. Since

$$E[(Y_j-\theta Z_{j-1})^2|\mathfrak{F}_{j-1}]=\theta(1-\theta)Z_{j-1},$$

using the fact that for any sequence $\{a_i, i \geq 1\}$ such that $a_i \geq 0$ series $\sum_{i=1}^{\infty} a_i (\sum_{j=1}^{i} a_j)^{-2}$ is convergent whenever $\sum_{i=1}^{\infty} a_i = \infty$, we obtain

$$\sum_{j=1}^{\infty} \frac{E[(Y_j - \theta Z_{j-1})^2 | \mathfrak{F}_{j-1}]}{(\sum_{i=1}^{j} Z_{i-1})^2} < \infty.$$

Now the assertion of the Proposition 1 follows from Theorem B and equality

$$\frac{\sum_{i=1}^{n} Y_i}{\sum_{i=1}^{n} Z_{i-1}} - \theta = \frac{T_n}{\sum_{i=1}^{n} Z_{i-1}}.$$

Proposition 2. *If $m < 1$, $\lambda < \infty$ and $\sigma^2 < \infty$ then $\hat{m}_n \to m$ almost surely as $n \to \infty$.*

Remark. As it was mentioned before, it was shown in [8] that $\hat{m}_n$ is the maximum likelihood estimator for m, which, generally speaking, does not imply consistency of the estimator. Therefore here we establish the strong consistency of the estimator $\hat{m}_n$.

Proof. We consider

$$\hat{m}_n - m = \frac{N(n)}{D(n)},$$

where

$$N(n) = \sum_{i=1}^{n} \sum_{j=1}^{W_{i-1}} (X_{ij} - m).\ D(n) = \sum_{i=1}^{n} W_{i-1}$$

Let F_n for each $n \geq 1$ be σ-algebra generated by $\{W_i, i = 0, 1, ..., n\}$. Then it is it is not difficult to see that $\{N(n), F_n\}$ is a martingale and again we can use Theorem B. We easily find that

$$E\left[\left(\sum_{j=1}^{W_{i-1}} (X_{ij} - m)\right)^2 |F_{i-1}\right] = \sigma^2 W_{i-1}$$

Therefore the series

$$\sum_{i=1}^{\infty} \frac{E\left[\left(\sum_{j=1}^{W_{i-1}} (X_{ij} - m)\right)^2 |F_{i-1}\right]}{D^2(i)} = \sum_{i=1}^{\infty} \frac{\sigma^2 W_{i-1}}{D^2(i)}$$

is convergent. Thus we conclude due to Theorem B that $N(n)/D(n)$ a.s. converges to zero as $n \to \infty$.

Proof of Theorem 1. Part (a) of the theorem is a direct consequence of Proposition 1 and consistency of Pakes estimator $\hat{\mu}_n$.

We now prove part (b). We represent $\hat{t}_n - \theta\mu$ as following:

$$\hat{t}_n - \theta\mu = I_1(n) + I_2(n), \tag{4.2}$$

where

$$I_1(n) = \theta(\frac{1}{n}\sum_{i=0}^{n-1} Z_i - \mu),\ I_2(n) = \frac{1}{n}(\sum_{i=1}^{n} Y_i - \theta \sum_{i=0}^{n-1} Z_i).$$

We immediately obtain from Theorem 3 in [9] that $\sqrt{n} I_1(n)$ as $n \to \infty$ converges in distribution to a normal random variable V_1 with mean zero and variance $\theta(\mu\sigma^2 + \gamma^2)/(1-m)^2$.

We now consider $I_2(n)$. It is not difficult to see that

$$\sqrt{n} I_2(n) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \sum_{j=1}^{Z_{i-1}} \overline{\xi}_{i-1j}, \tag{4.3}$$

where $\overline{\xi}_{ij} = \xi_{ij} - \theta$ is a centered Bernoulli random variable for each $(i, j) \in \mathcal{N}^2$. We apply Theorem A to the sum on the right side of (4.3). Let $\mathscr{A}_{ij}$ be the sigma-algebra generated by $\{X_{ij}^1, X_{ij}^2, \xi_{ij}\}$ for all $(i, j) \in \mathcal{N}^2$, $\mathscr{A}_0 = \{\varnothing, \Omega\}$ and

$$F_{kj} = \prod_{i=1}^{k-1} \prod_{l=1}^{Z_{i-1}} \mathscr{A}_{i-1l} \times \prod_{l=1}^{j} \mathscr{A}_{k-1l} \times \mathscr{A}_0.$$

It is clear that $\{Z_{k-1} > j\} \in F_{k,j}$ for any j which means that assumption on the number of summands in Theorem A is fulfilled. Using notation of the Theorem A we find that

$$\theta_{kj} = E[\bar{\xi}_{k-1j}|F_{kj-1}] = E[\bar{\xi}_{ij}] = 0,\ \sigma^2_{kj} = E[\bar{\xi}^2_{k-1j}|F_{kj-1}] = E[\bar{\xi}^2_{ij}] = \theta(1-\theta).$$

To check condition (3.4), using the fact that $\hat{\mu}_n \to \mu$ a.s. as $n \to \infty$, we obtain that

$$\frac{1}{n}\sum_{k=1}^{n}\sum_{j=1}^{Z_{k-1}}\sigma^2_{kj} = \frac{\theta(1-\theta)}{n}\sum_{k=0}^{n-1}Z_k \xrightarrow{P} \theta(1-\theta)\mu$$

as $n \to \infty$. Therefore, for any $\varepsilon > 0$

$$\lim_{n\to\infty} P\left\{\frac{1}{n}\sum_{k=1}^{n}\sum_{j=1}^{Z_{k-1}}\sigma^2_{kj} > \theta(1-\theta)\mu(1+\varepsilon)\right\} = 0,$$

which means that condition (3.4) is satisfied.

Since $\{\xi_{ij}, (i,j) \in \mathcal{N}^2\}$ are i.i.d. with zero mean, conditions (3.5) and (3.6) are also trivially satisfied. To check condition (3.7), we denote

$$K_n(x) =: \sum_{k=1}^{n}\sum_{j=1}^{Z_{k-1}} E[\frac{\bar{\xi}^2_{k-1j}}{n}\chi(\frac{\bar{\xi}_{k-1j}}{\sqrt{n}} \le x)|F_{kj-1}]$$

Since

$$K_n(x) = \frac{1}{n}E[\bar{\xi}^2_{11}\chi(\frac{\bar{\xi}_{11}}{\sqrt{n}} \le x)]\sum_{k=0}^{n-1}Z_k,$$

we easily find that as $n \to \infty$

$$K_n(x) = \xrightarrow{P} K(x) := \begin{cases} \theta(1-\theta)\mu, & \text{if } x \ge 0, \\ 0, & \text{if } x < 0. \end{cases}$$

Thus we obtain from Theorem A that $\sqrt{n}I_2(n) \xrightarrow{d} V_2$ as $n \to \infty$, where

$$Ee^{itV_2} = \exp\left\{-\frac{t^2\theta(1-\theta)\mu}{2}\right\},$$

which means that $V_2 \sim \sqrt{\theta(1-\theta)\mu}N(0,1)$, where $N(0,1)$ is the standard normal random variable.

Proof of Theorem 2. First we prove consistency of the estimator $\hat{a}_n$. We represent it as following

$$\hat{a}_n = A_1(n) + A_2(n) + A_3(n), \tag{4.4}$$

where

$$A_1(n) = \frac{\sum_{i=1}^{n} Z_i}{\sum_{i=1}^{n} Y_i}\left(\frac{\sum_{i=2}^{n+1} Y_i}{\sum_{i=1}^{n} Z_i} - \theta\right),$$

$$A_2(n) = \frac{\theta\sum_{i=1}^{n}(Z_i - \nu_i)}{\sum_{i=1}^{n} Y_i},\quad A_3(n) = \frac{\sum_{i=1}^{n}(\theta\nu_i - \eta_{i+1})}{\sum_{i=1}^{n} Y_i}$$

It follows from Proposition 1 that $A_1(n) \to 0$ a.s. as $n \to \infty$. Now we consider

$$A_3(n) = \frac{\sum_{i=1}^{n}\sum_{j=1}^{\nu_i}(\theta - \zeta_{ij})}{\sum_{i=1}^{n} Y_i}.$$

Since due to strong low of large numbers

$$\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{\nu_i}(\theta - \zeta_{ij}) \to E[\sum_{j=1}^{\nu_1}(\theta - \zeta_{1j})] = 0$$

a.s. as $n \to \infty$, using Part (a) of Theorem 1 we conclude that $A_3(n)$ a.s. converges to zero as $n \to \infty$. What concerns $A_2(n)$ we can rewrite it as following

$$A_2(n) = \frac{\theta \sum_{i=1}^n Z_{i-1}}{\sum_{i=1}^n Y_i} \cdot \frac{\sum_{i=1}^n (Z_i - \nu_i)}{\sum_{i=1}^n Z_{i-1}}.$$

From here, using Proposition 1 and consistency of Nanthi estimator $\hat{m}_n$, we obtain that $A_2(n) \to m$ a.s. as $n \to \infty$. Part (a) of Theorem 2 now follows from equality (4.4).

We now prove part (b). For this we consider

$$B(n) := (\hat{a}_n - m) \sum_{i=1}^n Y_i. \tag{4.5}$$

It is not difficult to see that $B(n)$ can be represented as following

$$B(n) = \sum_{i=2}^{n+1} (Y_i - \eta_i - mY_{i-1}) = \sum_{i=2}^{n+1} (Y_i - \theta Z_{i-1}) + \theta \sum_{i=2}^{n+1} (Z_{i-1} - mZ_{i-2} - \nu_{i-1})$$

$$+\theta \sum_{i=2}^{n+1} (\nu_{i-1} - \frac{\eta_i}{\theta}) + m \sum_{i=2}^{n+1} (\theta Z_{i-2} - Y_{i-1}) = B_1(n) + B_2(n) + B_3(n) + B_4(n).$$

It is obvious that

$$B_1(n) = \sum_{i=2}^{n+1} \sum_{j=1}^{Z_{i-1}} \bar{\xi}_{ij},$$

where $\{\bar{\xi}_{ij}, (i,j) \in \mathcal{N}^2\}$ are defined in (4.3). Therefore we obtain from the proof of Theorem 1 that $n^{-1/2} B_1(n) \xrightarrow{d} N(0, \mu(1-\theta)\theta)$ as $n \to \infty$, where $N(0, a^2)$ is a normal random variable with mean zero and the variance a^2. Similarly we can see that

$$B_4(n) = -m \sum_{i=2}^{n+1} \sum_{j=1}^{Z_{i-2}} \bar{\xi}_{ij},$$

and, therefore, the same arguments as in the proof of Theorem 1 allows us to conclude that $n^{-1/2} B_4(n) \xrightarrow{d} N(0, m^2 \mu(1-\theta)\theta)$ as $n \to \infty$.

We now consider $B_2(n)$. It is easy to see that it can be presented as following

$$B_2(n) = \theta \sum_{i=2}^{n+1} Z_{i-2} \left[\frac{\sum_{i=2}^{n+1} (Z_{i-1} - \nu_{i-1})}{\sum_{i=2}^{n+1} Z_{i-2}} - m \right].$$

Therefore, if we use Nanthi theorem , we obtain that as $n \to \infty$

$$\frac{\sqrt{n}}{\theta \sum_{i=2}^{n+1} Z_{i-2}} B_2(n) \xrightarrow{d} N(0, \sigma^2/\mu). \tag{4.6}$$

What concerns $B_3(n)$, we can rewrite it as following

$$B_3(n) = \sum_{i=2}^{n+1} \sum_{i=1}^{\nu_{i-1}} (\theta - \zeta_{i-1j}).$$

It is not difficult to see that

$$\sum_{i=1}^{\nu_{i-1}} (\theta - \zeta_{i-1j}), i = 2, 3, \ldots$$

are independent and identically distributed random variables with mean zero and variance $\Delta^2 =: \lambda \theta (1-\theta) + (\gamma\theta)^2$. Therefore it follows from the central limit theorem that $n^{-1/2} B_3(n)$ as $n \to \infty$ converges in distribution to $N(0, \Delta^2)$ a normal random variable with mean zero and the variance Δ^2.

Now we can rewrite (4.5) as following

$$\sqrt{n}(\hat{a}_n - m) = \frac{1}{\hat{t}_n \sqrt{n}} B(n).$$

If we take into account that $\hat{t}_n \to \theta\mu$ almost surely as $n \to \infty$ and the above results related to $B_i(n)$, $i = 1, 2, 3, 4$, we obtain the assertion of Part (b) of Theorem 2.

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References

[1] Andresson H., Britton T. (2000). *Stochastic Epidemic Models and their Statistical Analysis*, Springer, Ser. LN in Statistics 151, New York.
[2] Hall, P., Heyde, C. C. (1980). *Martingale Limit Theory and Its Application.* Wiley, New York.
[3] Heyde, C., C., Seneta, E. (1972). Estimation theory for growth and immigration rates in a multiplicative process. *J. Appl. Probab.* **9** 235-258.
[4] Heyde, C., C., Seneta, E. (1974). Notes on ”Estimation theory for growth and immigration rates in a multiplicative process.” *J. Appl. Probab.* **11** 572-577.
[5] Kvitkovicvova A., Panaretos V., M.(2011) Asymptotic inference for partially observed branching processes, Adv. Appl. Probab. 43, 1166-1190.
[6] Meester R., Trapman P. (2006) Estimation in branching processes with restricted observations, Adv. Appl. Probab., 38, 1098-1115.
[7] Meester R., De Koning J., De Jong M., S., Diekmann O. (2002) Modeling and real-time prediction of classical swine fever epidemics, Biometrics, 58, 178-184.
[8] Nanthi K. (1979). Some limit theorems of statistical relevance on branching processes. Ph.D. thesis submitted to the University of Madras.
[9] Pakes A. G. (1971). Branching processes with immigration. J. Appl. Probability, 8, 32-42.
[10] Panaretos V., M.(2007) Partially observed branching processes for stochastic epidemics, J. Math. Biol., 54, 645-668.
[11] Rahimov I. (1995). *Random Sums and Branching Stochastic Processes*. Springer, LNS 96. New York.
[12] Scott D. J. (1978). A central limit theorem for martingales and an application to branching processes. Stoch. Process. Appl. 6, 241-252.
[13] Venkataraman K. N., Nanthi K. (1982). A limit theorem on subcritical Galton-Watson process with immigration. The Ann. Probab., 10, No 4, 1069-1074.