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Generalized Order Statistics from Chen Distribution and its Characterization

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Abstract: In this paper, we have established the recurrence relations for single and product moments of generalized order statistics from Chen distribution. The result includes as particular cases of recurrence relations for moments of order statistics, sequential order statistics, progressive type II censored order statistics and k^{th} records. Further, using the recurrence relation for single moments, we have obtained the characterization result for Chen distribution.

Keywords: Generalized order statistics, order statistics, upper records, recurrence relations, Chen distribution, characterization.

1 Introduction

The concept of generalized order statistics (gos) was introduced by [7] as below: Let $F()$ be an absolutely continuous distribution function (df) with probability density function (pdf) $f()$. Further, let $n \in \mathbb{N}$, $k > 0$, $\tilde{m} = (m_1, m_2, ..., m_{n-1}) \in \mathbb{R}^{n-1}$, $M_r = \sum_{j=r}^{n-1} m_j$, such that $\gamma_r = k + n - r + M_r > 0$, for all $r \in 1, 2, ..., n-1$. Then $X(1, n, \tilde{m}, k), X(2, n, \tilde{m}, k), \ldots, X(n, n, \tilde{m}, k)$ are said to be the gos if their joint pdf is given by

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}[\bar{F}(x_i)]^{m_i} f(x_i)\right)[\bar{F}(x_n)]^{k-1} f(x_n) \tag{1.1}$$

on the cone $F^{-1}(0+) < x_1 \leq x_2 \leq ... \leq x_n < F^{-1}(1)$ of $\mathbb{R}^n$. Here $\bar{F}(x) = 1 - F(x)$ denotes the survival function.
Choosing the parameters appropriately, models such as ordinary order statistics
$(m = 0, k = 1 \; i.e. \; \gamma_i = n - i + 1)$, k^{th} record value $(m = -1, \; k \in \mathbb{N} \; i.e. \; \gamma_i = k)$, sequential order statistics $[\gamma_i = (n - i + 1)\beta_i; \beta_1, \beta_2, ..., \beta_n > 0]$, order statistics with non integral sample size $[\gamma_i = (\beta - i + 1); \beta > 0]$, Pfeifer record values $(\gamma_i = \beta_i; \beta_1, \beta_2, ..., \beta_n > 0)$ and progressive type II censored order statistics $(m \in \mathbb{N}, k \in \mathbb{N})$ can be obtained as particular cases of gos. For simplicity we have assumed that $m_1 = m_2 = ... = m_{n-1} = m$.

The pdf of r^{th} gos is given by [7]

$$f_{X(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}(\bar{F}(x))^{\gamma_r - 1} g_m^{r-1}(F(x)) f(x), \quad -\infty \leq x \leq \infty \tag{1.2}$$

and the joint pdf of $X(r, n, m, k)$ and $X(s, n, m, k)$, $1 \leq r < s \leq n$ is given by

$$f_{X(r,n,m,k),X(s,n,m,k)}(x,y) = \frac{C_{s-1}}{(r-1)!(s-r-1)!}[\bar{F}(x)]^m g_m^{r-1}(F(x))$$

$$\times [h_m(F(y)) - h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s - 1} f(x) f(y), \quad -\infty \leq x < y \leq \infty \tag{1.3}$$

where

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$$C_{r-1}=\prod_{i=1}^{r}\gamma_i,\ \ h_m(x)=\begin{cases}-\frac{(1-x)^{m+1}}{m+1}, & m\neq -1\\ -\ln(1-x), & m=-1\end{cases}$$

and

$$g_m(x)=h_m(x)-h_m(0),\quad x\in(0,1).$$

Recurrence relations are quite useful in computing the moments. The result given in present paper can be used to compute the moments of order random variables if the parent population follows Chen distributions. There are sufficient literature available regarding the recurrence relations of distributions based on *gos* and the characterization result based on these recurrence relation. For the review of literature, one may refer to [1,3,8,9,10]. For the textbook reference, the readers are referred to [2,4,7]. In this paper, we have established the recurrence relations for single and product moments of *gos* from the Chen distribution. Results for order statistics and k^{th} upper record values are also deduced. Further, characterization of Chen distribution based on recurrence relation for single moments of *gos* is also investigated.

A random variable X is said to have the Chen distribution if its probability density function (pdf) is of the form

$$f(x)=\lambda\beta x^{\beta-1}e^{x^{\beta}}\exp[\lambda(1-e^{x^{\beta}})],\quad x,\lambda,\beta>0, \tag{1.4}$$

and the distribution function (df) is given by

$$F(x)=1-\exp[\lambda(1-e^{x^{\beta}})],\quad x,\lambda,\beta>0. \tag{1.5}$$

From equation(1.5) and (1.6), the relation between pdf and df is given by

$$\bar{F}(x)=\frac{x^{1-\beta}e^{-x^{\beta}}}{\lambda\beta}f(x) \tag{1.6}$$

The Chen distribution given in (1.5) was introduced by [5]. This is a two-parameter lifetime distribution with bathtub shape or increasing failure rate function.The corresponding failure rate function $h(x)$ of this distribution may have a bathtub shape when $\beta<1$ and the distribution has increasing failure rate function when $\beta\geq 1$. At $\beta=1$, Chen distribution reduces to Gompertz distribution having *pdf*

$$f(x)=\lambda e^{x}\exp(\lambda[1-e^{x})],\quad x,\lambda>0 \tag{1.7}$$

The paper is divided into four section. In the section 2, we have obtained the recurrence relation for single moments of *gos* for Chen distribution. In section 3, we have deduced the recurrence relation for product moments of *gos* for Chen distribution. In section 4, the characterization result based on the recurrence relation for single moments of *gos* for Chen distribution is given.

2 Recurrence relation for single moments of *gos* from Chen distribution

In this section, the recurrence relation for single moments of *gos* from Chen distribution has been deduced. Further, the recurrence relation for single moments of order statistics and record values are obtained as a particular cases of *gos*.

Theorem 2.1: Let X be a non-negative continuous random variable and follows Chen distribution given in (1.5). Suppose that for any $j>0$ and $1\leq r\leq n$, $E|[\phi(X(r,n,m,k))]|$ is finite, then

$$E[X^j(r,n,m,k)]-E[X^j(r-1,n,m,k)]=\frac{j}{\lambda\beta\gamma_r}E[\phi(X(r,n,m,k))], \tag{2.1}$$

where $\phi(x)=x^{j-\beta}e^{-x^{\beta}}$.

Proof. We have

$$E[X^j(r,n,m,k)]=\frac{C_{r-1}}{(r-1)!}\int_0^{\infty}x^j[\bar{F}(x)]^{\gamma_r-1}g_m^{r-1}[F(x)]f(x)dx$$

Integrating by parts taking $[\bar{F}(x)]^{\gamma_r-1}f(x)$ as the part to be integrated, we get

$$E[X^j(r,n,m,k)] = \frac{jC_{r-1}}{(r-1)!\gamma_r}\int_0^\infty x^{j-1}[\bar{F}(x)]^{\gamma_r} g_m^{r-1}[F(x)]dx$$
$$+\frac{\gamma_r C_{r-2}}{\gamma_r(r-2)!}\int_0^\infty x^j[\bar{F}(x)]^{\gamma_{r-1}-1} g_m^{r-2}[F(x)]f(x)dx$$

which implies that

$$E[X^j(r,n,m,k)] - E[X^j(r-1,n,m,k)] = \frac{jC_{r-1}}{(r-1)!\gamma_r}\int_0^\infty x^{j-1}[\bar{F}(x)]^{\gamma_r} g_m^{r-1}[F(x)]dx \tag{2.2}$$

Now in view of equation (1.7), we have

$$E[X^j(r,n,m,k)] - E[X^j(r-1,n,m,k)]$$
$$= \frac{jC_{r-1}}{(r-1)!\gamma_r\lambda\beta}\int_0^\infty x^{j-\beta}e^{-x^\beta}[\bar{F}(x)]^{\gamma_r-1} g_m^{r-1}[F(x)]f(x)dx$$

Thus we get,

$$E[X^j(r,n,m,k)] - E[X^j(r-1,n,m,k)] = \frac{j}{\lambda\beta\gamma_r}E[\phi(X(r,n,m,k))]$$

and hence the result.

Remark 2.1: At $\beta = 1$ in (2.1), we get the recurrence relation for single moments of generalized order statistics from Gompertz distribution.
Remark 2.2: Putting $m = 0$ and $k = 1$ in (2.1), we obtain a recurrence relation for single moments of order statistics of the Chen distribution as

$$E[X^j_{r:n}] - E[X^j_{r-1:n}] = \frac{j}{\lambda\beta(n-r+1)}E[\phi(X_{r:n})] \tag{2.3}$$

Remark 2.3: Putting $m = -1$ and $k \geq 1$ in (2.1), we get a recurrence relation for single moments of k^{th} upper record values from Chen distribution as

$$E[X^j(r,n,-1,k)] - E[X^j(r-1,n,-1,k)] = \frac{j}{\lambda\beta k}E[\phi(X(r,n,-1,k))] \tag{2.4}$$

3 Recurrence relation for product moments of *gos* from Chen distribution

In this section, the recurrence relation for product moments of *gos* from Chen distribution has been deduced. Further, the recurrence relation for product moments of order statistics and record values are obtained as a particular cases of *gos*.

Theorem 3.1: Let X be a non-negative continuous random variable and follows Chen distribution given in (1.5). Suppose $E|[\phi\{X(r,n,m,k)X(s,n,m,k)\}]|$ is finite for any $i, j > 0$ and $1 \leq r < s \leq n$, then the recurrence relation for product moment is

$$E[X^i(r,n,m,k)X^j(s,n,m,k)] - E[X^i(r,n,m,k)X^j(s-1,n,m,k)] \tag{3.1}$$
$$= \frac{j}{\gamma_s\lambda\beta}E[\psi\{X(r,n,m,k)X(s,n,m,k)\}]$$

where $\psi(x,y) = x^i y^{j-\beta}e^{-y^\beta}$.

Proof. In view of (1.3), we have

$$E[X^i(r,n,m,k)X^j(s,n,m,k)] = \frac{C_{s-1}}{(r-1)!(s-r-1)!}\int_0^\infty x^i[\bar{F}(x)]^m f(x)g_m^{r-1}[F(x)]I(x)dx$$

where $I(x) = \int_x^\infty y^j[h_m(F(y)) - h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s-1}f(y)dy$

solving the integral $I(x)$ by parts and substituting the resulting expression in (3.2), we get

$$E[X^i(r,n,m,k)X^j(s,n,m,k)] - E[X^i(r,n,m,k)X^j(s-1,n,m,k)]$$
$$= \frac{jC_{s-1}}{(r-1)!(s-r-1)!\gamma_s}\int_0^\infty \int_x^\infty x^i[\bar{F}(x)]^m f(x) g_m^{r-1}(F(x)) y^{j-1}$$
$$\times [h_m(F(y)) - h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s-1}\frac{y^{1-\beta}e^{-y^\beta}}{\lambda\beta} f(y)dydx$$

this implies that

$$E[X^i(r,n,m,k)X^j(s,n,m,k)] - E[X^i(r,n,m,k)X^j(s-1,n,m,k)]$$
$$= \frac{j}{\gamma_s\lambda\beta}E[\psi\{X(r,n,m,k)X(s,n,m,k)\}]$$

and hence the result.

Remark 3.1: At $\beta = 1$ in (3.1), we get the recurrence relation for product moments of generalized order statistics from Gompertz distribution.

Remark 3.2: Putting $m = 0$ and $k = 1$ in (3.1), we obtain recurrence relations for product moments of order statistics of the Chen distribution as

$$E[X_{r:n}^i X_{s:n}^j] - E[X_{r:n}^i X_{s-1:n}^j] = \frac{j}{\lambda\beta(n-s+1)}E[\psi(X_{r:n}X_{s:n})] \tag{3.2}$$

Remark 3.3: Putting $m = -1$ and $k \geq 1$ in (3.1), we get the recurrence relations for product moments of upper k^{th} record values from Chen distribution in the form

$$E[X^i(r,n,-1,k)X^j(s,n,-1,k)] - E[X^i(r,n,-1,k)X^j(s-1,n,-1,k)] \tag{3.3}$$
$$= \frac{j}{\lambda\beta k}E[\psi\{X(r,n,-1,k)X(s,n,-1,k)\}]$$

4 Characterization of Chen distribution

In this section, applying the generalization of the M*ü*ntz-Sz*á*sz Theorem [6], we have established a characterization result of Chen distribution based on single moments of *gos*.

Theorem 4.1: For $m > -1$, the necessary and sufficient condition for a random variable X to be distributed with pdf given in (1.6) is that

$$\frac{j}{\lambda\beta\gamma_r}E[\phi(X(r,n,m,k))] = E[X^j(r,n,m,k)] - E[X^j(r-1,n,m,k)] \tag{4.1}$$

if and only if

$$F(x) = 1 - \exp(\lambda(1-e^{x^\beta})), \quad x,\lambda,\beta > 0.$$

Proof: A necessary part follows immediately from (4.1). On the other hand if the recurrence relation (4.1) is satisfied, then

$$\frac{jC_{r-1}}{(r-1)!\gamma_r\lambda\beta}\int_0^\infty \phi(x)[\bar{F}(x)]^{\gamma_r-1}g_m^{r-1}[F(x)]f(x)dx$$
$$= \frac{C_{r-1}}{(r-1)!}\int_0^\infty x^j[\bar{F}(x)]^{\gamma_r-1}g_m^{r-1}[F(x)]f(x)dx - \frac{C_{r-1}}{(r-1)!}\frac{(r-1)}{\gamma_r}\int_0^\infty x^j[\bar{F}(x)]^{\gamma_r+m}g_m^{r-2}[F(x)]f(x)dx$$
$$= \frac{C_{r-1}}{(r-1)!}\int_0^\infty x^j[\bar{F}(x)]^{\gamma_r}g_m^{r-2}[F(x)]f(x)\left[\frac{g_m[F(x)]}{[F(x)]} - \frac{(r-1)[\bar{F}(x)]^m}{\gamma_r}\right]dx$$

Let

$$v(x) = -\frac{[\bar{F}(x)]^{\gamma_r}g_m^{r-1}[F(x)]}{\gamma_r},$$

then

$$v'(x) = [\bar{F}(x)]^{\gamma_r} g_m^{r-2}[F(x)] f(x) \left[\frac{g_m[F(x)]}{[F(x)]} - \frac{(r-1)[\bar{F}(x)]^m}{\gamma_r} \right]$$

Thus

$$\frac{jC_{r-1}}{(r-1)!\gamma_r \lambda \beta} \int_0^\infty \phi(x)[\bar{F}(x)]^{\gamma_r - 1} g_m^{r-1}[F(x)] f(x) dx = \frac{C_{r-1}}{(r-1)!} \int_0^\infty x^j v'(x) dx \tag{4.2}$$

Now integrating RHS of (4.2) by parts and using the value of $v(x)$, we get

$$\frac{jC_{r-1}}{(r-1)!\gamma_r \lambda \beta} \int_0^\infty \phi(x)[\bar{F}(x)]^{\gamma_r - 1} g_m^{r-1}[F(x)] f(x) dx = \frac{C_{r-1}}{(r-1)!\gamma_r} \int_0^\infty j x^{j-1} [\bar{F}(x)]^{\gamma_r} g_m^{r-1}[F(x)] dx$$

which reduces to

$$\frac{jC_{r-1}}{(r-1)!\gamma_r} \int_0^\infty [\bar{F}(x)]^{\gamma_r} g_m^{r-1}[F(x)] \left[x^{j-1} - \frac{\phi(x) f(x)}{\lambda \beta F(x)} \right] dx = 0 \tag{4.3}$$

Applying a generalization of the Müntz-Szász Theorem [6] to
equation(4.5), which states that on a space $L(a,b)$ of all summable functions defined on the interval (a,b), a sequence of functions $f_n(x)$ is complete on (a,b) if for any $g \in L(a,b)$ the equalities

$$\int_a^b f_n(x) g(x) dx = 0 \quad n = 1, 2, \ldots,$$

imply that $g(x) = 0$ almost everywhere on (a,b), then we get

$$\frac{\bar{F}(x)}{f(x)} = \frac{x^{1-\beta} e^{-x^\beta}}{\lambda \beta},$$

which implies that

$$F(x) = 1 - \exp(\lambda(1 - e^{x^\beta})), \quad x, \lambda, \beta > 0.$$

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References

[1] Ahmad, A. A. and Fawzy, M. (2002): Recurrence relations for single moments of generalized order statistics from doubly truncated distributions. *J. Stat. Plann. Infer.*, **117(2)**, 241-249.
[2] Ahsanullah, M. and Nevzorov, V. B. (2001): *Ordered Random Variables*. Nova, New York.
[3] AL-Hussaini, E. K., Ahmad, A. A. and AL-Kashif, M. A. (2005): Recurrence relations for moment and conditional moment generating functions of generalized order statistics, *Metrika*, **61**, 199-220.
[4] Arnold, B. C., Balakrishnan, N. and Nagaraja, H. N. (1992): *A First Course in Order Statistics*. Wiley, New York.
[5] Chen, Z. (2000): A new two-parameter lifetime distribution with bathtub shape or increasing failure rate function. *Statist. Probab. Letters*, **49**, 155-161.
[6] Hwang, J. S. and Lin, G. D. (1984): On a generalized moments problem II. *Proc. Amer. Math. Soc.*, **91**, 577-580.
[7] Kamps, U. (1995): *A Concept of Generalized Order Statistics*. B. G. Teubner Stuttgart.
[8] Kumar, D. and Khan, M. I. (2013): Relations for Generalized Order Statistics from Doubly Truncated Generalized Exponential Distribution and its Characterization. *Math. Sci. Letters*, **2**, 9-18.
[9] Kumar, D. (2015): On the quotient moment of lower generalized order statistics and characterization. *J. Applied Math. Statist. Informatics*. **11(1)**, 73-89.
[10] Pawlas, P. and Szynal, D. (2001): Recurrence relations for single and product moments of k^{th} record values from Pareto, Generalized Pareto and Burr distributions. *Commun. Statist. Theory Math.*, **30(4)**, 1699-1709.

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