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# Fast Evaluation of Canonical Oscillatory Integrals

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**Abstract:** Fast evaluation of oscillatory integrals is an issue attracts much attention in many fields. In this paper, we are interested in the calculation of canonical oscillatory integrals, and the irregular oscillatory integrals are transformed into canonical ones with respect to the presences of stationary phase points or not. An improved-Levin method is proposed to calculated the canonical oscillatory integral, where the eigen-decomposition is employed to solve the target system of linear equation, and a much higher efficiency is yielded in comparison with the direct solution methods such as the Gaussian elimination.

**Keywords:** Oscillatory integral, Stationary phase point, Clenshaw-Curtis method, Improved-Levin method, Chebyshev spectral method, Eigen-decomposition.

## 1. Introduction

How to calculate the oscillatory integrals like

$$I[f] = \int_a^b f(x)\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x, \quad \omega \gg 1, \tag{1}$$

is a key issue arising in many fields[1]. In this kind of integral, $f(x)$ and $g(x)$ are both smooth and non-oscillatory, and they are called the amplitude function and the phase function, respectively. During the past decades, some special quadrature methods for oscillatory integrals have been developed, and the representative ones include the asymptotic expansion method[2,3], the Filon(-type) method[4, 5], the Levin(-type) method[6,7], the (numerical-)steepest descent method[8], the stationary phase point method[9], and others. Among them the Filon method and the Levin method have attracted much attention. The Filon(-type) method has a relatively long history but it is generally only applicable for oscillatory integrals whose moments $\int \phi_k(x)\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x$ are explicitly known ($\phi_k$ are the basis functions), so its generality is limited. The Levin method has better applicability for oscillatory integrals with complicated phase functions, but the system of linear equations involved tends to be ill-conditioned and the integral result is susceptible to the ill-conditioning. J. Li and etc. have put forward an improved-Levin method to remove the impact of the ill-conditioning in [10–12], where the well-conditioned system of linear equations is solved by the Gaussian elimination method, and the ill-conditioned one is solved by the TSVD (truncated singular value decomposition) method. During the processing, the computational complexity of Gaussian elimination method is $\mathcal{O}((N+1)^3)$ ($N+1$ is the number of nodes used in the calculation), and the TSVD method requires even more operations. Although the iterative method can cut down the computational complexity to $\mathcal{O}((N+1)^2)$ in each iteration step, the total computational complexity differs for different oscillatory integrals due to the different convergence rate of them. Later, Olver introduced a GMRES-Levin method to calculate the oscillatory integrals in [13], where the differentiations involved are calculated via FFT and the total operations are believed to be on the level of $\mathcal{O}((N+1)\log(N+1))$. However, as stated in [14], the FFT has a lot of nested operations which create overhead that is not apparent in the count of multiplications and additions. In fact, the FFT is generally competitive only for very large $N$. For moderate $N$, the direct matrix multiplication is actually even more efficient. In the filed of fast computation of oscillatory integrals, the main idea is to obtain the integral result using small or moderate number of nodes, rather than very large number of nodes, so it is useful to find other approaches to proceed the computations via matrix multiplications.

In this paper, we transform the irregular oscillatory integrals into canonical ones, and then propose an improved-

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Levin method to compute the transformed integrals. During the process of the improved-Levin method, the eigen-decomposition is employed to solve the target system of linear equations, which is finally boiled down to two matrix-multiplication process and the operation counts can be reduced from $\mathcal{O}((N+1)^3)$ (for traditional methods such as the Gaussian elimination) to $\mathcal{O}(2(N+1)^2)$.

## 2. Transformation of an oscillatory integral from irregular form to canonical form

In this paper, we are interested in the numerical evaluation of canonical highly oscillatory integrals of the form

$$I[f] = \int_{-1}^{1} f(x)\mathrm{e}^{\mathrm{i}\omega x}\mathrm{d}x. \tag{2}$$

As will be shown in the next section, oscillatory integrals of this canonical type can be efficiently calculated. But for an irregular oscillatory integral with complex phase function, the situation is a little complicated. This section will show that it can also be transformed into a canonical form. The derivations are proceed in two cases: those free of stationary phase points and those with.

### *2.1. The case free of stationary phase points*

In this case, we assume that the oscillatory integral is in the form of (1) and the phase function $g(x)$ is free of stationary phase point in the interval $[a,b]$ (i.e., $\forall x \in [a,b], g'(x) \neq 0$). Let $y = g(x)$, then the oscillator in (1) can be transformed into a canonical one[15]:

$$I[f] = \int_{g(a)}^{g(b)} \frac{f(g^{-1}(y))}{g'(g^{-1}(y))}\mathrm{e}^{\mathrm{i}\omega y}\mathrm{d}y. \tag{3}$$

Moreover, introducing the coordinate transform $y = v(x) = \frac{g(b)-g(a)}{2}x + \frac{g(b)+g(a)}{2}$ to (3) yields an oscillatory over a normalized interval:

$$I[f] = \int_{-1}^{1} F(x)\mathrm{e}^{\mathrm{i}\xi x}\mathrm{d}x, \tag{4}$$

where $F(x) = \frac{g(b)-g(a)}{2}\mathrm{e}^{\mathrm{i}\omega\frac{g(a)+g(b)}{2}}\frac{f(g^{-1}(v(x)))}{g'(g^{-1}(v(x)))}$ is non-singular, and $\xi = \frac{g(b)-g(a)}{2}\omega$. In this transformation, one requires for knowing the inverse of $g(x)$. The inverse can generally be easily worked out using Newton's method[15], but in many practical applications, it even can be analytically obtained.

### *2.2. The case with stationary phase points*

The stationary phase point of an oscillatory integral is a point where the first derivative of the phase function vanishes (i.e., $\exists x \in [a,b], g'(x) = 0$). It is known that a stationary phase point plays a very important roll in determining the integral result[16], so the integrand around it should be carefully handled. In this paper we assume that the stationary phase points are located at the endpoints. This assumption is justified since an oscillatory integral with stationary phase points inside the interval can be subdivided into several sub-integrals according to the positions of the stationary phase points:

$$\begin{aligned} I[f] &= \left(\int_a^{c_1} + \int_{c_1}^{c_2} + \cdots + \int_{c_m}^{b}\right) f(x)\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x \\ &\triangleq \sum_{n=0}^{m} I_n[f], \end{aligned} \tag{5}$$

where $c_1 < c_2 < \cdots < c_m$ are $m$ stationary phase points.

Consider one of the integrals on the right hand side of (5). For convenience the lower and upper limits of the integral are denoted by $c$ and $d$, respectively. As shown in (5), the stationary phase points of this integral are located at the endpoints, so $g'(x) \neq 0$ holds for $x \in (c,d)$. However, while transforming this integral to (3) to obtain a canonical form, the presence of stationary phase points will introduce singularity to the amplitude function $\frac{f(g^{-1}(y))}{g'(g^{-1}(y))}$ because $g'(x)$ vanishes at the endpoints. In order to avoid this difficult, special transformation should be considered.

Let $h(x) = \int_c^x f(t)\mathrm{d}t$, then the concerned sub-integral is transformed as

$$I_n[f] = h(x)\mathrm{e}^{\mathrm{i}\omega g(x)}\Big|_c^d - \mathrm{i}\omega\int_c^d h(x)\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}g(x). \tag{6}$$

In the open interval $(c,d)$ the phase function $g(x)$ is monotonic and invertible because the stationary phase points only exist at the endpoints. In this sense, introducing the transformation $y = g(x)$ to (6) yields

$$I_n[f] = h(x)\mathrm{e}^{\mathrm{i}\omega g(x)}\Big|_c^d - \mathrm{i}\omega\int_{g(c)}^{g(d)} h(g^{-1}(y))\mathrm{e}^{\mathrm{i}\omega y}\mathrm{d}y. \tag{7}$$

In principle, the amplitude function $h(g^{-1}(y))$ should be non-singular because both $f(t)$ and $h(x) = \int_c^x f(t)\mathrm{d}t$ are non-singular. However, this function could be very sharp due to the presence of stationary phase points at the endpoints, and this will then introduce difficulty to the evaluation of $\int_{g(c)}^{g(d)} h(g^{-1}(y))\mathrm{e}^{\mathrm{i}\omega y}\mathrm{d}y$.

Moreover, in order to overcome the problem caused by the sharps, an additional transformation is proposed. Let $k(y) = \int_{g(c)}^{y} h(g^{-1}(u))\mathrm{d}u$, then the integral (7) can be transformed as follows by using the integration by parts:

$$I_n[f] = \mathrm{e}^{\mathrm{i}\omega g(d)}\left[h(d) - \mathrm{i}\omega k(g(d))\right] - \omega^2\int_{g(c)}^{g(d)} k(y)\mathrm{e}^{\mathrm{i}\omega y}\mathrm{d}y, \tag{8}$$

where we have considered $h(c) = 0$ and $k(g(c)) = 0$.

According to the definitions of $h(x)$ and $k(y)$ we have

$$k(y) = \int_{g(c)}^{y}\left(\int_c^{g^{-1}(u)} f(t)\mathrm{d}t\right)\mathrm{d}u. \tag{9}$$

then substituting the assumption $u = g(s)$ to (9) gives

$$k(y) = \int_c^{g^{-1}(y)} \int_c^s f(t)g'(s)\mathrm{d}t\mathrm{d}s.$$

This is a 2D non-oscillatory integral because $f(x)$ and $g(x)$ are both non-oscillatory, so the function $k(y)$ can be easily calculated by the Gauss-type quadrature algorithms[17]. At the same time, the derivative of $k(y)$ is determined as

$$k'(y) = \frac{\mathrm{d}\int_{g(c)}^{y} h(g^{-1}(t))\mathrm{d}t}{\mathrm{d}y} = h(g^{-1}(y)),$$

so the plot of $k(y)$ should be very smooth since the function value of $h(x)$ is limited.

At the same time, the oscillatory integral in (8) can be transformed into a canonical one if the coordinate transforation $y = \frac{g(d)-g(c)}{2}t + \frac{g(d)+g(c)}{2}$ is applied to it. Therefore, in the following study, we pay the main attention to the calculation of canonical oscillatory integrals like (2).

## 3. Evaluation of canonical oscillatory integrals

An early attempt on the fast evaluation of canonical oscillatory integrals was made by Filon [4] who obtained the integral result by approximating the amplitude function with a polynomial and computing the moment $\int x^k \mathrm{e}^{\mathrm{i}\omega x}\mathrm{d}x$ analytically. The work was later improved by Clenshaw-Curtis[21], Luke[20], Bakhvalov[22], Patterson[23], and some others[15, 18, 19]. Among them, the Clenshaw-Curtis method has attracted much attention for its good stability. This method works well on the canonical oscillatory integrals. It uses FFT to save the calculation cost of obtaining the coefficients $\alpha_k^N[f]$, but its profit is greatly diminished when a moderate or small $N$ is taken into account. In the following section, we would like to introduce an alternative evaluation method for canonical oscillatory integrals based on the Levin method. It will be shown that the method has a comparable accuracy to the Clenshaw-Curtis method, and it is also with a very high efficiency.

According to Levin's theory, to solve the integral (2) is to solve an ordinary differential equation (ODE) without boundary condition:

$$p'(x) + \mathrm{i}\omega p(x) = f(x). \tag{10}$$

If a proper function $p(x)$ is solved from the ODE, then the integral result is obtained as

$$I = p(1)\mathrm{e}^{\mathrm{i}\omega} - p(-1)\mathrm{e}^{-\mathrm{i}\omega}. \tag{11}$$

Due to the good stability of Chebyshev expansion, we use the Chebyshev collocation spectral method to solve the ordinary differential equation. At the same time, the Chebyshev-Lobatto nodes cluster around the endpoints, so it is advisable to use these nodes since the integrand around the endpoint plays an important role in determining the integral result.

The differentiation matrix is a powerful tool in the spectral method[24, 25, 14, 26, 27]. Applying the differentiation matrix $\mathbf{D}$ to the ODE (10) yields the following system of linear equations:

$$(\mathbf{D} + \mathrm{i}\omega)\,\mathbf{p} = \mathbf{f}, \tag{12}$$

where $\mathbf{p}$ and $\mathbf{f}$ are two numerical vectors corresponding to the unknown function values and the amplitude function values, respectively. This system of linear equations can be solved by traditional methods such as the Gaussian elmination method and TSVD method[11, 10], but the operations required by these routines are $\mathcal{O}((N+1)^3)$ or even more. Here we would like to introduce another solution scheme for (12), and the count of operations can be reduced from $\mathcal{O}((N+1)^3)$ to $\mathcal{O}(2(N+1)^2)$.

Let the eigen-decomposition of matrix $\mathbf{D}$ be

$$\mathbf{DV} = \mathbf{V}\mathbf{\Lambda}, \tag{13}$$

where $\mathbf{V}$ is the matrix whose $k^{\text{th}}$ column is the eigenvector $\mathbf{v}_k$ of $\mathbf{D}$ and $\mathbf{\Lambda}$ is the diagonal matrix whose diagonal elements are the corresponding eigenvalues, i.e., $\mathbf{\Lambda}_{kk} = \lambda_k$. In this manner, the target system of linear equations (12) can be re-expressed as

$$\mathbf{V}(\mathbf{\Lambda} + \mathrm{i}\omega\mathbf{E})\mathbf{V}^{-1}\mathbf{p} = \mathbf{f} \tag{14}$$

with $\mathbf{E}$ being an identity matrix. Introducing $\tilde{\mathbf{p}} = \mathbf{V}^{-1}\mathbf{p}$ and $\tilde{\mathbf{f}} = \mathbf{V}^{-1}\mathbf{f}$ to (14) yields

$$(\mathbf{\Lambda} + \mathrm{i}\omega\mathbf{E})\tilde{\mathbf{p}} = \tilde{\mathbf{f}}. \tag{15}$$

Then the solution of (15) can be directly obtained as

$$\tilde{\mathbf{p}}_k = \frac{\tilde{\mathbf{f}}_k}{\lambda_k + \mathrm{i}\omega}, \tag{16}$$

and the solution of (12) is finally obtained as $\mathbf{p} = \mathbf{V}\tilde{\mathbf{p}}$. According to the numerical vector $\mathbf{p}$, the integral result is yielded as

$$I[f] \sim \mathbf{p}(1)\mathrm{e}^{\mathrm{i}\omega} - \mathbf{p}(N+1)\mathrm{e}^{-\mathrm{i}\omega}. \tag{17}$$

It is noted that the differentiation matrices $\mathbf{D}$ for different $N$ are all known, so the eigen-decompositions of them (corresponding to $\mathbf{V}$ and $\mathbf{\Lambda}$) can be pre-computed. In this sense, the operations of the above-mentioned processes mainly come from the the following two operations: $\tilde{\mathbf{f}} = \mathbf{V}^{-1}\mathbf{f}$ and $\mathbf{p} = \mathbf{V}\tilde{\mathbf{p}}$.

Let the triangle decomposition of $\mathbf{V}$ be

$$\mathbf{TV} = \mathbf{LU}, \tag{18}$$

where $\mathbf{T}$ is a permutation matrix, $\mathbf{L}$ is a lower triangle matrix and $\mathbf{U}$ is an upper triangle matrix, then the result

$\tilde{\mathbf{f}} = \mathbf{V}^{-1}\mathbf{f}$ is determined by solving the following two systems of linear equations:

$$\mathbf{LX} = \mathbf{Tf}, \quad \mathbf{U}\tilde{\mathbf{f}} = \mathbf{X}, \tag{19}$$

where $\mathbf{X}$ is a temporary numerical vector. It is known that $\mathbf{L}$ and $\mathbf{U}$ are both triangle matrices and can be pre-computed, so the computational complexity of each equation in (19) is about $\mathcal{O}((N+1)^2/2)$, resulting a total computational complexity of $\mathcal{O}((N+1)^2)$ for $\tilde{\mathbf{f}} = \mathbf{V}^{-1}\mathbf{f}$. At the same time, the matrix-multiplication $\mathbf{p} = \mathbf{V}\tilde{\mathbf{p}}$ is with a computational complexity of $\mathcal{O}((N+1)^2)$. In this manner, the total computational complexity of the present method is $\mathcal{O}(2(N+1)^2)$, which is much smaller than $\mathcal{O}((N+1)^3)$ for the direct solution methods (such as Gaussian elimination).

## 4. Numerical examples

*Example 1*.Calculate the integral

$$I = \int_{-1}^{1} [\sin x + \cos(7x)]\, \mathrm{e}^{\mathrm{i}\omega(x-2)^2} \mathrm{d}x.$$

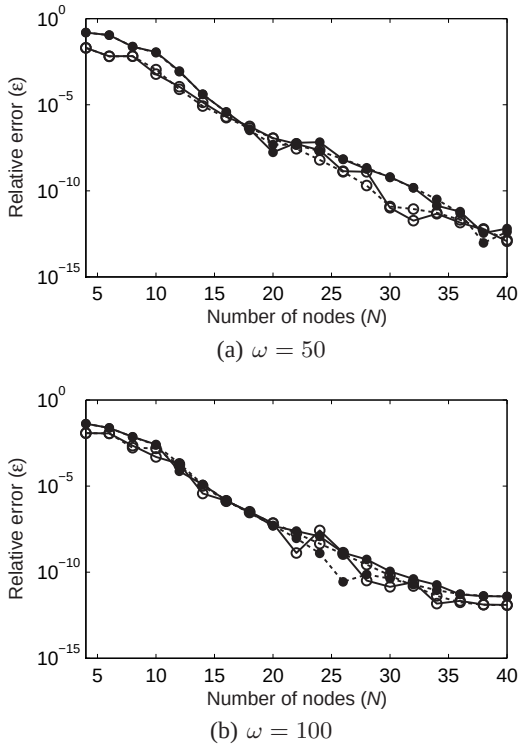
For this integral, no stationary phase point is involved, so it can be directly transformed into the canonical form of (2). The performances of the proposed methods (the Clenshaw-Curtis method and the improved-Levin method) on different number of nodes ($N = 2, 4, \cdots, 40$) are measured by the relative error $\varepsilon$. If the numerical integral result is $I' = \Re(I') + \mathrm{i}\Im(I')$ and the exact integral value is $I = \Re(I) + \mathrm{i}\Im(I)$, then the real and imaginary part of the relative error are defined as $\varepsilon_{\mathrm{r}} = \frac{\Re(I'-I)}{\Re(I)}$ and $\varepsilon_{\mathrm{i}} = \frac{\Im(I'-I)}{\Im(I)}$, respectively.

Figure 1 presents the relative errors of the the Clenshaw-Curtis method and the improved-Levin method. Form this figure it is observed that the two methods have comparable performances, and the relative error curves descend with the increase of the number of nodes. These phenomena can be explained as follows:

1. More nodes can give better interpolation of the amplitude function and phase function, so the relative error curves descend with the increase of the number of nodes.
2. The amplitude functions in both the two methods are interpolated by the same nodes (Chebyshev-Lobatto nodes), and the additional errors, corresponding to computing the moments $I_k(T_k, \omega)$ and solving the system of linear equations (12), are very small, so the accuracy of the two methods should be on the same level for fixed number of nodes.

*Example 2*.Calculate the integral

$$I = \int_{-1}^{1} (x^2 + x)\mathrm{e}^{\mathrm{i}\omega\sqrt{1+(x+1)^2}} \mathrm{d}x.$$




(a) $\omega = 50$

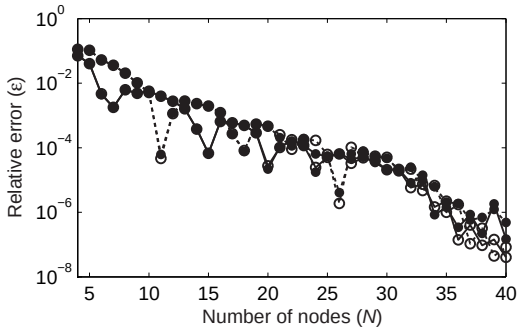
(b) $\omega = 100$

**Figure 1** The relative errors in approximating integral $\int_{-1}^{1} [\sin x + \cos(7x)]\, \mathrm{e}^{\mathrm{i}\omega(x-2)^2} \mathrm{d}x$ by the Clenshaw-Curtis method $(\cdots)$ and the improved-Levin method (—) for two choices of $\omega$, including the real part (•) and the imaginary part (○).

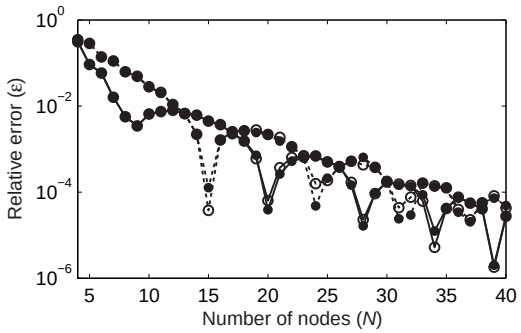
This kind of oscillator is very common in the field of electromagnetics. For this integral, a single stationary phase point $x = -1$ is involved, so it should be transformed following the way mentioned in section 2.2. During this comparison, the number of nodes $N$ varies as $N = 4, 5, \cdots, 40$, and the relative errors of the Clenshaw-Curtis method and the improved-Levin method are presented in Figure 2. It is also seen that the two methods have comparable accuracy, and more nodes correspond to higher accuracies.

## 5. Conclusion

This paper provides an approach of transforming an irregular oscillatory integral into a canonical one, and presents an improved-Levin quadrature method to calculate the canonical oscillatory integral. In the method, the eigen-decomposition is adopted to solve the system of linear equations, and a much higher efficiency is achieved in comparison with conventional method such as the Gaussian elimination.




(a) $\omega = 50$




(b) $\omega = 100$

**Figure 2** The relative error in approximating integral $\int_{-1}^{1}(x^2+x)\mathrm{e}^{\mathrm{i}\omega\sqrt{1+(x+1)^2}}\mathrm{d}x$ by the Clenshaw-Curtis method $(\cdots)$ and the improved-Levin method (—) for two choices of $\omega$, including the real part (•) and the imaginary part (○).

## References


[1] J. Li, X. Wang, T. Wang, *Progress in Electromagnetics Research* **107**, 219 (2010).
[2] A. Iserles, S. P. Nørsett, *BIT Numerical Mathematics* **46**, 549 (2006).
[3] A. Iserles, S. P. Nørsett, *Math. Comp.* **75**, 1233 (2006).
[4] L. N. G. Filon, *Proceedings of the Royal Society of Edinburgh* **49**, 38 (1928).
[5] A. Iserles, S. P. Nørsett, *Proceedings Royal Soc.* **461**, 1383 (2005).
[6] D. Levin, *Mathematics of Computation* **38**, 531 (1982).
[7] S. Olver, *IMA Journal of Numerical Analysis* **26**, 213 (2006).
[8] D. Huybrechs, *SIAM J. Numer. Anal.* **44**, 1026 (2006).
[9] W. C. Chew, *Waves and fields in inhomogeneous media* (Van Nostrand Reinhold, New York, 1990).
[10] J. Li, X. Wang, T. Wang, *Science in China (series F)* **51**, 1614 (2008).
[11] J. Li, X. Wang, T. Wang, S. Xiao, *Applied Numerical and Mathematics* **60**, 833 (2010).
[12] J. Li, X. Wang, T. Wang, C. Shen, *Applied Mathematics and Computation* **209**, 327 (2009).
[13] S. Olver, *BIT Numerical Mathematics* **50**, 149 (2010).
[14] J. P. Boyd, *Chebyshev and Fourier Spectral Methods* (DOVER Publications, New York, 2000).
[15] G. A. Evans, J. R. Webster, *Journal of Computational and Applied Mathematics* **112**, 55 (1999).
[16] A. Iserles, S. P. Nørsett, *BIT Numerical Mathematics* **44**, 755 (2004).
[17] P. J. Davis, P. Rabinowitz, *Methods of Numerical Integration* (Academic Press, New York, 1984).
[18] S. Olver, Numerical approximation of highly oscillatory integrals, Phd thesis, University of Cambridge, Cambridge (2008).
[19] S. Xiang, *Journal of Computational and Applied Mathematics* (2006).
[20] Y. L. Luke, *Proc.Cambridge.Philos.Soc.* (Cambridge, 1954), vol. 50 of *Proc.Cambridge.Philos.Soc.*, pp. 269–277.
[21] C. W. Clenshaw, A. R. Curtis, *Numerische Mathematik* **2**, 197 (1960).
[22] N. S. Bakhvalov, L. G. Vasil'eva, *Comput.Math.Phys* **8**, 241 (1968).
[23] T. Patterson, *Numer. Math.* **27**, 42 (1976).
[24] L. N. Trefethen, *Spectral Methods in Matlab* (SIAM, Philadelphia, 2000).
[25] J. Shen, T. Tang, *Spectral and High-Order Methods with Applications* (Science Press, Beijing, 2006).
[26] C. Canuto, M. Y. Hussaini, A. Quarteroni, T. A. Zang, *Spectral methods in fluid dynamics* (Springer, New York, 1988).
[27] M. E. Elbarbary, M. El-Sayed, *Applied Numerical Mathtematics* **55**, 425 (2005).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