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On Characterizations of Relatively P– and P_0– Properties in Nonsmooth Functions

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Abstract: For H-differentiable function f from a closed rectangle Q in R^n into R^n, a result of Song, Gowda and Ravindran [On Characterizations of **P**- and $\mathbf{P}_0$-Properties in Nonsmooth Functions. Mathematics of Operations Research. 25: 400-408 (2000)] asserts that f is a $\mathbf{P}(\mathbf{P_0})-$ function on Q if the H_Q-differential $T_Q(x)$ at each $x \in Q$ consisting of $\mathbf{P}(\mathbf{P_0})-$ matrices. In this paper, we introduce the concepts of relatively $\mathbf{P}(\mathbf{P_0})-$ properties in order to extend these results to nonsmooth functions when the underlying functions are H-differentiable. We give characterizations of relatively $\mathbf{P}(\mathbf{P_0})-$ of vector nonsmooth functions. Also, our results give characterizations of relatively $\mathbf{P}(\mathbf{P_0})-$ when the underlying functions are C^1-functions, semismooth-functions, and for locally Lipschitzian functions. Moreover, we show useful applications of our results by giving illustrations to generalized complementarity problems.

Keywords: H-Differentiability, semismooth-functions, locally Lipschitzian, generalized Jacobian, $\mathbf{P}(\mathbf{P_0})$– properties, generalized complementarity problems.

1 Introduction

In this article, we give characterizations of relatively P– and P_0– properties in nonsmooth functions. For functions $f, g : Q \subseteq R^n \to R^n$, we say that f and g are relatively $\mathbf{P_0}(\mathbf{P})$-functions if, for any $x \neq y$ in R^n,

$$\max_{\{i: x_i \neq y_i\}} [f(x) - f(y)]_i [g(x) - g(y)]_i \geq 0 \ (>0). \tag{1}$$

If $g(x) = x$, then relatively $\mathbf{P_0}(\mathbf{P})$-functions reduce to $\mathbf{P_0}(\mathbf{P})$-function, i.e., we say that f is a $\mathbf{P_0}(\mathbf{P})$-function if, for any $x \neq y$ in Q,

$$\max_{\{i: x_i \neq y_i\}} (x-y)_i [f(x) - f(y)]_i \geq 0 \ (>0). \tag{2}$$

A matrix $M \in R^{n\times n}$ is said to be a $\mathbf{P}_0(\mathbf{P})$-matrix if the function $f(x) = Mx$ is a $\mathbf{P_0}(\mathbf{P})$-function or equivalently, every principle minor of M is nonnegative (respectively, positive [5]). $\mathbf{P}_0(\mathbf{P})$- functions have attracted many researchers in the areas of complementarity and (box) variational inequality problem, see e.g., [10] and the references therein. It is known that $\mathbf{P}_0(\mathbf{P})$-function is a generalization to every monotone (strictly monotone) function.

Fiedler and Pták in [6] showed that for an affine function $f(x) = Mx + q$, $\mathbf{P}_0(\mathbf{P})$–property on R^n is equivalent to M being $\mathbf{P}_0(\mathbf{P})$– matrix. for Fréchet differentiable function f on a closed rectangle Q, Gale and Nikaid [7] gave a characterization of **P**–property, i.e., if the Jacobian matrix $\nabla f(x)$ is a **P**– matrix at all $x \in Q$, then f is is a **P**– function. Subsequently, Moré and Rheinboldt [13] gave a characterization of $\mathbf{P_0}$–property, i.e., the Jacobian matrix $\nabla f(x)$ is a $\mathbf{P_0}$– matrix on Q if and only if f is is a $\mathbf{P_0}$– function on Q. In [19], Song, Gowda, and Ravindran extended above results and characterized $\mathbf{P}_0$– and **P**–properties when the underlying function is H-differentiable. Moreover, they illustrated these characterizations to nonlinear complementarity problems. Motivated from the above results, we raise the following questions: Can we extend the $\mathbf{P_0}(\mathbf{P})$-function in (2) in order to give general characterizations when the underlying functions are H-differentiable? Are these characterizations useful in the area of complementarity problems, and variational inequalities? This paper answers these questions. We introduce the concepts of

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relatively $\mathbf{P}(\mathbf{P_0})-$ properties and give characterizations of these concepts under appropriate conditions. Moreover, we show the usefulness of these characterizations by giving some illustrations to generalized complementarity problems when the underlying functions are C^1-functions, semismooth-functions, locally Lipschitzian functions, and H-differentiable.

2 Preliminaries

Throughout this paper, we regard vectors in R^n as column vectors. Vector inequalities are interpreted componentwise. For a set $K \subseteq R^n$, coK denotes the convex hull of K and $\overline{K}$ denotes the closure of K [17]. For a differentiable function $f : R^n \to R^m$, $\nabla f(\overline{x})$ denotes the Jacobian matrix of f at $\overline{x}$. For a matrix A, A_i denotes the ith row of A.

Definition 1. *A function $\phi : R^2 \to R$ is called a GCP function if $\phi(a,b) = 0 \Leftrightarrow ab = 0, a \geq 0, b \geq 0$. For the problem $GCP(f,g)$, we define*

$$\Phi(x) = \left[\, \phi(f_1(x),g_1(x)) \ldots \phi(f_i(x),g_i(x)) \ldots \phi(f_n(x),g_n(x)) \,\right]^T \tag{3}$$

and, we call $\Phi(x)$ a GCP function for $GCP(f,g)$.

2.1 H-differentiability and H-differentials

The concepts of H-differentiability and H-differentials were introduced in [9] to study the injectivity of nonsmooth functions. It has been shown in [9] that the Fréchet derivative of a Fréchet differentiable function, the Clarke generalized Jacobian of a locally Lipschitzian function [1], the Bouligand subdifferential of a semismooth function [12], [14], [16], and the C-differential of a C-differentiable function [15] are examples of H-differentials. Any superset of an H-differential is an H-differential, H-differentiability implies continuity, and H-differentials enjoy simple sum, product and chain rules, see [9]. The H-differentiable function need not be locally Lipschitzian nor directionally differentiable [22]. These concepts give useful and unified treatments for many problems in optimization, complementarity problems, and variational inequalities when the underlying functions are not necessarily locally Lipschitzian nor semismooth, see [8], [9], [19], [20], [21], [22], [23], [24], [25], [26], [27] .

We now recall the following from Gowda and Ravindran [9].

Definition 2. *Given a function $F : \Omega \subseteq R^n \to R^m$ where Ω is an open set in R^n and $x^* \in \Omega$, we say that a nonempty subset $T(x^*)$ (also denoted by $T_F(x^*)$) of $R^{m\times n}$ is an H-differential of F at x^* if for every sequence $\{x^k\} \subseteq \Omega$ converging to x^*, there exist a subsequence $\{x^{k_j}\}$ and a matrix $A \in T(x^*)$ such that*

$$F(x^{k_j}) - F(x^*) - A(x^{k_j} - x^*) = o(||x_j^k - x^*||). \tag{4}$$

We say that F is H-differentiable at x^* if F has an H-differential at x^*.

Remark. It is shown that in [24] if a function $F : \Omega \subseteq R^n \to R^m$ is H-differentiable at a point $\overline{x}$, then there exist a constant $L > 0$ and a neighbourhood $B(\overline{x},\delta)$ of $\overline{x}$ with

$$||F(x) - F(\overline{x})|| \leq L||x - \overline{x}||, \quad \forall x \in B(\overline{x},\delta). \tag{5}$$

Conversely, if condition (5) holds, then $T(\overline{x}) := R^{m\times n}$ can be taken as an H-differential of F at $\overline{x}$. We thus have, in (5), an alternate description of H-differentiability.
Clearly any function locally Lipschitzian at $\overline{x}$ will satisfy (5). For real valued functions, condition (5) is known as the 'calmness' of F at $\overline{x}$. This concept has been well studied in the literature of nonsmooth analysis (see [18], Chapter 8).

The rest of this section shows the Fréchet derivative of a Fréchet differentiable function, the Clarke generalized Jacobian of a locally Lipschitzian function, the Bouligand subdifferential of a semismooth function, and the C-differential of a C-differentiable function are particular instances of H-differentials [9].

Fréchet differentiable functions

Let $F : R^n \to R^m$ be Fréchet differentiable at $x^* \in R^n$ with Fréchet derivative matrix (= Jacobian matrix derivative) $\{\nabla F(x^*)\}$ such that

$$F(x) - F(x^*) - \nabla F(x^*)(x - x^*) = o(||x - x^*||).$$

Then F is H-differentiable with $\{\nabla F(x^*)\}$ as an H-differential.

Locally Lipschitzian functions

Let $F : \Omega \subseteq R^n \to R^m$ be locally Lipschitzian at each point of an open set Ω. For $x^* \in \Omega$, define the Bouligand subdifferential of F at x^* by

$$\partial_B F(x^*) = \{\lim \nabla F(x^k) : x^k \longrightarrow x^*, x^k \in \Omega_F\}$$

where Ω_F is the set of all points in Ω where F is Fréchet differentiable. Then, the (Clarke) generalized Jacobian [1]

$$\partial F(x^*) = co\partial_B F(x^*)$$

is an H-differential of F at x^*.

Semismooth functions

Consider a locally Lipschitzian function $F : \Omega \subseteq R^n \to R^m$ that is semismooth at $x^* \in \Omega$ [12], [14], [16]. This means for any sequence $x^k \to x^*$, and for $V_k \in \partial F(x^k)$,

$$F(x^k) - F(x^*) - V_k(x^k - x^*) = o(||x^k - x^*||).$$

Then the Bouligand subdifferential

$$\partial_B F(x^*) = \{\lim \nabla F(x^k) : x^k \longrightarrow x^*, x^k \in \Omega_F\}.$$

is an H-differential of F at x^*. In particular, this holds if F is piecewise smooth, i.e., there exist continuously differentiable functions $F_j : R^n \to R^m$ such that

$$F(x) \in \{F_1(x), F_2(x), \ldots, F_J(x)\} \quad \forall x \in R^n.$$

C-differentiable functions

Let $F : R^n \to R^n$ be C-differentiable [15] in a neighborhood D of x^*. This means that there is a compact upper semicontinuous multivalued mapping $x \mapsto T(x)$ with $x \in D$ and $T(x) \subset R^{n\times n}$ satisfying the following condition at any $a \in D$: For $V \in T(x)$,

$$F(x) - F(a) - V(x-a) = o(||x-a||).$$

Then, F is H-differentiable at x^* with $T(x^*)$ as an H-differential.
Remark The following simple example, is taken from [22], shows that an H-differentiable function need not be locally Lipschitzian nor directionally differentiable. Consider on R,

$$F(x) = x\sin(\frac{1}{x}) \text{ for } x \neq 0 \text{ and } F(0) = 0.$$

Then F is H-differentiable on R with

$$T(0) = [-1,1] \text{ and } T(c) = \{\sin(\frac{1}{c}) - \frac{1}{c}cos(\frac{1}{c})\} \text{ for } c \neq 0.$$

We note that F is not locally Lipschitzian around zero. We also see that F is neither Fréchet differentiable nor directionally differentiable.

3 The relatively $\mathbf{P}(\mathbf{P_0})$– properties in nonsmooth functions

The following result [13], [19] will be useful in this paper.

Theorem 1. *$f : R^n \to R^n$ is a $\mathbf{P}_0(\mathbf{P})$-function, under each the following conditions.*

(a) f is Fréchet differentiable on R^n and for every $x \in R^n$, the Jacobian matrix $\nabla f(x)$ is a $\mathbf{P}_0(\mathbf{P})$-matrix.
(b) f is locally Lipschitzian on R^n and for every $x \in R^n$, the generalized Jacobian $\partial f(x)$ consists of $\mathbf{P}_0(\mathbf{P})$-matrices.
(c) f is semismooth on R^n (in particular, piecewise affine or piecewise smooth) and for every $x \in R^n$, the Bouligand subdifferential $\partial_B f(x)$ consists of $\mathbf{P}_0(\mathbf{P})$-matrices.
(d) f is H-differentiable on R^n and for every $x \in R^n$, an H-differential $T_f(x)$ consists of $\mathbf{P}_0(\mathbf{P})$-matrices.

Remark. The converse statements in Theorem 1 are usually false for $\mathbf{P}$-conditions.

For $\mathbf{P_0}$-conditions in Theorem 1, the converse statements of Item (a) and Item (c) are true, while the converse statements of Item (b) and Item (d) may not hold in general, see [13] and [19].

The following Lemma is needed in our subsequent analysis.

Lemma 1. *Suppose $f, g : R^n \to R^n$ and g is one-to-one and onto. Define $h : R^n \to R^n$ where $h := f \circ g^{-1}$. Then f and g are relatively $\mathbf{P_0}(\mathbf{P})$-functions if and only if h is $\mathbf{P_0}(\mathbf{P})$-function.*

Proof.
Suppose f and g are relatively $\mathbf{P_0}(\mathbf{P})$– function, we need to show h is $\mathbf{P_0}(\mathbf{P})$ function. Since g is one-to-one and onto, for all $x, y \in R^n$, there exist unique $x^*, y^* \in R^n$ with $x = g^{-1}(x^*)$ and $y = g^{-1}(y^*)$. For all $x^* \neq y^* \in R^n$, we have

$$\begin{aligned}[h(x^*) - h(y^*)]_i^T [x^* - y^*]_i &= [h(g(x)) - h(g(y))]_i^T [g(x) - g(y)]_i \\ &= [f(x) - f(y)]_i^T [g(x) - g(y)]_i \\ &\geq 0.\end{aligned} \tag{6}$$

the converse follows a similar argument. □

The following proposition is given in [19] for $\mathbf{P}$– matrices.

Proposition 1. *Let $h : \Omega \to R^n$ be continuous where Ω is open set in R^n and H-differentiable at each point $\bar{x} \in \Omega$ with an H-differential $T_h(\bar{x})$ consisting of $\mathbf{P}$– matrices. Then there exists vectors u and v arbitrarily close to zero such that*

(i) $u < 0$ and $h(\bar{x} + u) < h(\bar{x})$;
(i) $v > 0$ and $h(\bar{x} + v) > h(\bar{x})$.

We recall that a continuous mapping is called a homeomorphism if it is a one-to-one and onto mapping and if its inverse mapping is also continuous.
The proof of the following theorem based on Proposition 1, is similar to the proofs of Theorem 3.4 in [11] and Theorem 1 in [19].

Theorem 2. *Let Q be a rectangular in R^n. Suppose $f : Q \to R^n$ and $g : Q \to R^n$ are continuous and H-differentiable at $\bar{x}$ with H-differentials, respectively, by $T_f(\bar{x})$ and $T_g(\bar{x})$. Suppose g is a homeomorphism. Let $h : Q \to R^n$ be continuous where $h := f \circ g^{-1}$ and H-differentiable at each point $\bar{x} \in \Omega$ with an H-differential $T_h(\bar{x})$ consisting of $\mathbf{P}$– matrices. Then h is a $\mathbf{P}$– function on Q, in particular, one-to-one. Moreover, f and g are relatively $\mathbf{P}$-functions.*

In view of Lemma 1 and Theorem 1, we have the following.

Corollary 1. *Let Q be a rectangular in R^n. Suppose $f : Q \to R^n$ and $g : Q \to R^n$ are continuous and H-differentiable at $\bar{x}$ with H-differentials, respectively, by $T_f(\bar{x})$ and $T_g(\bar{x})$. Suppose g is a homeomorphism. Let $h : Q \to R^n$ be continuous where $h := f \circ g^{-1}$ and H-differentiable at each point $\bar{x} \in \Omega$ with an H-differential $T_h(\bar{x})$ consisting of $\mathbf{P}-$ matrices. Then f and g are relatively $\mathbf{P}$-functions.*

Remark. Note that if $g(x) = x$ in Corollary 1, we get Theorem 1 in [19].

The following theorem characterizes the relatively $\mathbf{P_0}-$ property via H-differentials.

Theorem 3. *Let Q be a rectangular in R^n. Suppose $f : Q \to R^n$ and $g : Q \to R^n$ are continuous and H-differentiable at $\bar{x}$ with H-differentials, respectively, by $T_f(\bar{x})$ and $T_g(\bar{x})$. g is a homeomorphism. Let $h : Q \to R^n$ be continuous where $h := f \circ g^{-1}$ and H-differentiable at each point $\bar{x} \in \Omega$ with an H-differential $T_h(\bar{x})$ consisting of $\mathbf{P_0}-$ matrices. Then h is a $\mathbf{P_0}-$ function on Q.*

Remark. Note that if $g(x) = x$ in Theorem 2, we get Theorem 2 in [19].

In view of Lemma 1 and Theorem 2, we have the following.

Corollary 2. *Let Q be a rectangular in R^n. Suppose $f : Q \to R^n$ and $g : Q \to R^n$ are continuous and H-differentiable at $\bar{x}$ with H-differentials, respectively, by $T_f(\bar{x})$ and $T_g(\bar{x})$. g is a homeomorphism. Let $h : Q \to R^n$ be continuous where $h := f \circ g^{-1}$ and H-differentiable at each point $\bar{x} \in \Omega$ with an H-differential $T_h(\bar{x})$ consisting of $\mathbf{P_0}-$ matrices. Then f and g are relatively $\mathbf{P_0}$-functions.*

When f and g are Fréchet differentiable in which case $T_f(\bar{x}) = \{\nabla f(\bar{x})\}$ and $T_g(\bar{x}) = \{\nabla g(\bar{x})\}$, we have the following.

Corollary 3. *Let Q be a rectangular in R^n. Suppose $f : Q \to R^n$ and $g : Q \to R^n$ are continuous and Fréchet differentiable on R^n and for every $x \in R^n$, Fréchet derivatives (= Jacobians), respectively, by $\nabla f(x)$ and $\nabla g(x)$. Suppose that g is a homeomorphism. Let $h : Q \to R^n$ be continuous where $h := f \circ g^{-1}$ and Fréchet differentiable at each point $\bar{x} \in Q$ with with Fréchet derivative matrix (= Jacobian matrix derivative) $\nabla h(\bar{x})$ consisting of $\mathbf{P_0}-$ matrix. Then*

(i) h is a $\mathbf{P_0}-$ function on Q.
(ii) f and g are relatively $\mathbf{P_0}$-functions.

In view of Subsection 2.1, we have the following corollaries.

Corollary 4. *Let Q be a rectangular in R^n. Suppose $f : Q \to R^n$ and $g : Q \to R^n$ are continuous and locally Lipschitzian and for every $x \in R^n$, generalized Jacobians, respectively, by $\partial f(\bar{x})$ and $\partial g(\bar{x})$. suppose that g is a homeomorphism. Let $h : Q \to R^n$ be continuous where $h := f \circ g^{-1}$ and locally Lipschitzian at each point $\bar{x} \in Q$ with generalized Jacobian $\partial h(\bar{x})$ consisting of $\mathbf{P_0}-$ matrices. Then*

(i) h is a $\mathbf{P_0}-$ function on Q.
(ii) f and g are relatively $\mathbf{P_0}$-functions.

Corollary 5. *Let Q be a rectangular in R^n. Suppose $f : Q \to R^n$ and $g : Q \to R^n$ are continuous and semismooth on R^n (in particular, piecewise affine or piecewise smooth) and for every $x \in R^n$, the Bouligand subdifferentials, respectively, by $\partial_B f(\bar{x})$ and $\partial_B g(\bar{x})$. suppose that g is a homeomorphism. Let $h : Q \to R^n$ be continuous where $h := f \circ g^{-1}$ and semismooth (in particular, piecewise affine or piecewise smooth) at each point $\bar{x} \in Q$ with Bouligand subdifferential $\partial_B h(\bar{x})$ consisting of $\mathbf{P_0}-$ matrices. Then*

(i) h is a $\mathbf{P_0}-$ function on Q.
(ii) f and g are relatively $\mathbf{P_0}$-functions.

When $g(x) = x$ in the above results, we get the following.

Corollary 6. *Under each of the following, $f : R^n \to R^n$, is $\mathbf{P}(\mathbf{P_0})-$ function.*

(a) f is Fréchet differentiable on R^n and for every $x \in R^n$, the Jacobian matrix $\nabla f(x)$ is a $\mathbf{P}(\mathbf{P_0})-$ matrix.
(b) f is locally Lipschitzian on R^n and for every $x \in R^n$, the generalized Jacobian $\partial f(x)$ consists of $\mathbf{P}(\mathbf{P_0})-$ matrices.
(c) f is semismooth on R^n (in particular, piecewise affine or piecewise smooth) and for every $x \in R^n$, the Bouligand subdifferential $\partial_B f(x)$ consists of $\mathbf{P}(\mathbf{P_0})-$ matrices.

4 Some applications to generalized complementarity problems

In this section, we present some applications to generalized complementarity problems to illustrate the usefulness of our results. We consider a generalized complementarity problem, denoted by GCP(f,g), which is to find a vector $\bar{x} \in R^n$ such that

$$f(\bar{x}) \geq 0,\ g(\bar{x}) \geq 0 \text{ and } \langle f(\bar{x}), g(\bar{x})\rangle = 0$$

where $f : R^n \to R^n$ and $g : R^n \to R^n$ are H-differentiable functions.

By considering an GCP function $\Phi : R^n \to R^n$ associated with GCP(f,g) and the corresponding merit function $\Psi = \frac{1}{2}||\Phi||^2$ when the underlying functions f and g are H-differentiable. It should be recalled that

$$\Psi(\bar{x}) = 0 \Leftrightarrow \Phi(\bar{x}) = 0 \Leftrightarrow \bar{x} \text{ solves GCP}(f,g).$$

We now give the following illustrations to show the usefulness of our results, but first we compute the H-differential of some GCP functions.

Example 1. The following function is called the penalized Fischer-Burmeister function [4]

$$\phi_\lambda(a,b) := \lambda\phi_{FB}(a,b) + (1-\lambda)a_+b_+ \tag{7}$$

where ϕ_{FB} is called Fischer-Burmeister function, $a_+ = \max\{0,a\}$ and $\lambda \in (0,1)$ is a fixed parameter. In this paper, we will consider the following GCP function:

$$\begin{aligned}\Phi(\bar{x}) &= \phi_\lambda(f(\bar{x}),g(\bar{x})) := \lambda\phi_{FB}(f(\bar{x}),g(\bar{x})) \\ &+(1-\lambda)f(\bar{x})_+g(\bar{x})_+.\end{aligned} \tag{8}$$

and its merit function associated to GCP function at $\bar{x}$ is

$$\Psi(\bar{x}) = \frac{1}{2}||\Phi(\bar{x})||^2$$

where

$$\begin{aligned}\Phi_i(\bar{x}) &= \phi_\lambda(f_i(\bar{x}),g_i(\bar{x})) := \lambda\phi_{FB}(f_i(\bar{x}),g_i(\bar{x})) \\ &+(1-\lambda)f_i(\bar{x})_+g_i(\bar{x})_+.\end{aligned} \tag{9}$$

Let $f : R^n \to R^n$ and $g : R^n \to R^n$ be H-differentiable functions at $\bar{x} \in R^n$ with H-differentials, $T_f(\bar{x})$ and $T_g(\bar{x})$, respectively.

A similar analysis can be carried out for the GCP function in (9). Let

$$\begin{aligned}J(\bar{x}) &= \{i : f_i(\bar{x}) = 0 = g_i(\bar{x})\} \text{ and} \\ K(\bar{x}) &= \{i : f_i(\bar{x}) > 0, g_i(\bar{x}) > 0\}.\end{aligned}$$

A straightforward calculation shows that the H-differential of Φ in (9) is given by

$$T_\Phi(\bar{x}) = \{VA+WB : (A,B,V,W,d) \in \Gamma\},$$

where Γ is the set of all quadruples (A,B,V,W,d) with $A \in T_f(\bar{x})$, $B \in T_g(\bar{x})$, $||d|| = 1$, $V = diag(v_i)$ and $W = diag(w_i)$ are diagonal matrices with

$$v_i = \begin{cases} \lambda\left(1-\frac{f_i(\bar{x})}{\sqrt{f_i(\bar{x})^2+g_i(\bar{x})^2}}\right)+(1-\lambda)g_i(\bar{x}) & \text{when } i \in K(\bar{x}) \\ \lambda\left(1-\frac{A_id}{\sqrt{(A_id)^2+(B_id)^2}}\right) & \text{when } i \in J(\bar{x}) \\ & \text{and } (A_id)^2+(B_id)^2 > 0 \\ \lambda\left(1-\frac{f_i(\bar{x})}{\sqrt{f_i(\bar{x})^2+g_i(\bar{x})^2}}\right) & \text{when } i \notin J(\bar{x}) \cup K(\bar{x}) \\ \text{arbitrary} & \text{when } i \in J(\bar{x}) \text{ and } (A_id)^2+(B_id)^2 = 0, \end{cases}$$

$$w_i = \begin{cases} \lambda\left(1-\frac{g_i(\bar{x})}{\sqrt{f_i(\bar{x})^2+g_i(\bar{x})^2}}\right)+(1-\lambda)f_i(\bar{x}) & \text{when } i \in K(\bar{x}) \\ \lambda\left(1-\frac{B_id}{\sqrt{(A_id)^2+(B_id)^2}}\right) & \text{when } i \in J(\bar{x}) \\ & \text{and } (A_id)^2+(B_id)^2 > 0 \\ \lambda\left(1-\frac{g_i(\bar{x})}{\sqrt{f_i(\bar{x})^2+g_i(\bar{x})^2}}\right) & \text{when } i \notin J(\bar{x}) \cup K(\bar{x}) \\ \text{arbitrary} & \text{when } i \in J(\bar{x}) \text{ and } (A_id)^2+(B_id)^2 = 0. \end{cases}$$

Example 2. Suppose that f and g are H-differentiable at $\bar{x}$ with H-differentials, respectively, by $T_f(\bar{x})$ and $T_g(\bar{x})$. Consider the following GCP function which is the basis of generalization of the Fischer-Burmeister function [2], [3]:

$$\phi_p(a,b) := a+b-\|(a,b)\|_p \tag{10}$$

where p is any fixed real number in the interval $(1,+\infty)$ and $\|(a,b)\|_p$ denotes the p-norm of (a,b), i.e., $\|(a,b)\|_p = \sqrt[p]{|a|^p+|b|^p}$.

Now we give the H-differentials of Φ_p. Let

$$J(\bar{x}) := \{i : f_i(\bar{x}) = 0 = g_i(\bar{x})\}.$$

The H-differential of Φ_p at $\bar{x}$ is given by

$$T_{\Phi_p}(\bar{x}) = \{VA+WB : (A,B,V,W,d) \in \Gamma\},$$

where Γ is the set of all quadruples (A,B,V,W,d) with $A \in T_f(\bar{x})$, $B \in T_g(\bar{x})$, $\|d\| = 1$, $V = \text{diag}(v_i)$ and $W = \text{diag}(w_i)$ are diagonal matrices satisfying the conditions:

$$|1-v_i|^{\frac{p}{p-1}}+|1-w_i|^{\frac{p}{p-1}} = 1, \quad \forall i = 1,2,\ldots,n,$$

and

$$v_i = \begin{cases} 1-\frac{|f_i(\bar{x})|^{p-1}\text{sgn}(f_i(\bar{x}))}{(|f_i(\bar{x})|^p+|g_i(\bar{x})|^p)^{\frac{p-1}{p}}} & i \notin J(\bar{x}), \\ 1-\frac{|A_id|^{p-1}\text{sgn}(A_id)}{(|A_id|^p+|B_id|^p)^{\frac{p-1}{p}}} & i \in J(\bar{x}) \text{ and } |A_id|^p+|B_id|^p > 0, \\ \text{arbitrary} & i \in J(\bar{x}) \text{ and } |A_id|^p+|B_id|^p = 0, \end{cases} \tag{11}$$

$$w_i = \begin{cases} 1-\frac{|g_i(\bar{x})|^{p-1}\text{sgn}(g_i(\bar{x}))}{(|f_i(\bar{x})|^p+|g_i(\bar{x})|^p)^{\frac{p-1}{p}}} & i \notin J(\bar{x}), \\ 1-\frac{|B_id|^{p-1}\text{sgn}(B_id)}{(|A_id|^p+|B_id|^p)^{\frac{p-1}{p}}} & i \in J(\bar{x}) \text{ and } |A_id|^p+|B_id|^p > 0, \\ \text{arbitrary} & i \in J(\bar{x}) \text{ and } |A_id|^p+|B_id|^p = 0. \end{cases} \tag{12}$$

Remark. The calculation in two illustration relies on the observation that the following is an H-differential of the one variable function $t \mapsto t_+$ at any $\bar{t}$:

$$\triangle(\bar{t}) = \begin{cases} \{1\} & \text{if } \bar{t} > 0 \\ \{0,1\} & \text{if } \bar{t} = 0 \\ \{0\} & \text{if } \bar{t} < 0. \end{cases}$$

In the following theorem we will minimize the merit function under P_0-conditions and the proof will be similar to Theorem 5 in [25].

Theorem 4. *Suppose $f : R^n \to R^n$ and $g : R^n \to R^n$ are H-differentiable at $\bar{x}$ with H-differentials, respectively, by $T_f(\bar{x})$ and $T_g(\bar{x})$. Suppose Φ is a GCP function of f and g. Assume that $\Psi := \frac{1}{2}||\Phi||^2$ is H-differentiable at $\bar{x}$ with an H-differential given by*

$$T_\Psi(\bar{x}) = \{\Phi(\bar{x})^T\,[VA+WB] : A \in T_f(\bar{x}), B \in T_g(\bar{x}), V = diag(v_i),$$
and
$$W = diag(w_i), \text{ with } v_iw_i > 0 \text{ whenever } \Phi_i(\bar{x}) \neq 0\}.$$

Further suppose that $T_g(\bar{x})$ consists of nonsingular matrices and $S(\bar{x})$ consists of $\mathbf{P}_0$-matrices where $S(\bar{x}) := \{AB^{-1} : A \in T_f(\bar{x}), B \in T_g(\bar{x})\}$. Then $0 \in T_\Psi(\bar{x}) \Leftrightarrow \Phi(\bar{x}) = 0$.

As consequences of the above result, one can state a general result for any GCP function, but for the sake of simplicity, we state the results for GCP function which is based on generalized Fischer-Burmeister function, $T_f(\bar{x}) = \{\nabla f(\bar{x})\}$ and $T_g(\bar{x}) = \{\nabla g(\bar{x})\}$.

Corollary 7. *Suppose $f : R^n \to R^n$ and $g : R^n \to R^n$ are Fréchet differentiable at $\bar{x}$. Suppose Φ is a GCP function of f and g, which is the basis of generalized Fischer-Burmeister function and $\Psi := \frac{1}{2}||\Phi||^2$. If $\nabla g(\bar{x})$ is a nonsingular matrix and the product $\nabla f(\bar{x})\nabla g(\bar{x})^{-1}$ is $\mathbf{P}_0$-matrix, then $\bar{x}$ is a local minimizer to Ψ if and only if $\bar{x}$ solves GCP(f,g).*

Corollary 8. *Suppose $f : R^n \to R^n$ and $g : R^n \to R^n$ are Fréchet differentiable at $\bar{x}$. Assume g is continuous, one-to-one, onto and $\nabla g(\bar{x})$ is a nonsingular matrix. Moreover, assume f and g are relatively $\mathbf{P_0}$-functions. Suppose Φ is a GCP function of f and g, which is based on generalized Fischer-Burmeister function and $\Psi := \frac{1}{2}||\Phi||^2$. Then $\bar{x}$ is a local minimizer to Ψ if and only if $\bar{x}$ solves GCP(f,g).*

Proof. Since g is a one-to-one and onto, and f and g are relatively $\mathbf{P_0}$-functions, by Lemma 1, the mapping $f \circ g^{-1}$ is $\mathbf{P_0}$-function which implies $\nabla f(\bar{x})\nabla g(\bar{x})^{-1}$ is $\mathbf{P}_0$-matrix. The proof follows from Corollary 7. □

It is known that a continuous, strongly monotone mapping $f : R^n \to R^n$ is a homeomorphism from R^n onto itself and the $\nabla f(\bar{x})$ is positive definite matrix if f is C^1 (see [13]). So we have the following.

Corollary 9. *Suppose $f : R^n \to R^n$ and $g : R^n \to R^n$ are Fréchet differentiable at $\bar{x}$. Assume g is continuous and strongly monotone. Moreover, assume f and g are relatively $\mathbf{P_0}$-functions. Suppose Φ is a GCP function of f and g, which is based on generalized Fischer-Burmeister function and $\Psi := \frac{1}{2}||\Phi||^2$. Then $\bar{x}$ is a local minimizer to Ψ if and only if $\bar{x}$ solves GCP(f,g).*

In view of subsection 2.1, Theorem 4, and Corollary 5, we get the following.

Corollary 10. *Suppose $f : R^n \to R^n$ and $g : R^n \to R^n$ are semismooth (piecewise smooth or piecewise affine) at $\bar{x}$ with Bouligand subdifferentials, respectively, by $\partial_B f(\bar{x})$ and $\partial_B g(\bar{x})$. Assume g is continuous, one-to-one, onto and $\partial_B g(\bar{x})$ consists of nonsingular matrices. Moreover, assume f and g are relatively $\mathbf{P_0}$-functions. Suppose Φ is a GCP function of f and g, which is based on generalized Fischer-Burmeister function and $\Psi := \frac{1}{2}||\Phi||^2$. Then $\bar{x}$ is a local minimizer to Ψ if and only if $\bar{x}$ solves GCP(f,g).*

Concluding Remarks Our goal of this article is to give characterization of relatively $\mathbf{P}(\mathbf{P_0})$-properties in nonsmooth functions when the underlying functions are H-differentiable.

We illustrate the usefulness of our results by applying these results a generalized complementarity problem corresponding to H-differentiable functions, with an associated GCP function Φ and a merit function $\Psi(x) = \frac{1}{2}||\Phi||^2$.

To the best of our knowledge, when the underlying functions are continuously differentiable (locally Lipschitzian, semismooth, and directionally differentiable) functions, our characterizations are new for relatively $\mathbf{P}(\mathbf{P_0})$-properties, and generalization to characterizations for $\mathbf{P}(\mathbf{P_0})$-properties. For example, when $g(x) = x$ and f is C^1 (in which case we can let $T_f(\bar{x}) = \{\nabla f(\bar{x})\}$), our characterizations of relatively $\mathbf{P}(\mathbf{P_0})$-properties reduce to characterization of $\mathbf{P}(\mathbf{P_0})$-properties in [19]. Moreover, GCP(f,g) reduces to nonlinear complementarity problem NCP(f) and the results of this paper will be valid for NCP(f) as applications. In view of Subsection 2.1, we have the following:

– If f and g are C^1 in which case $T_f(\bar{x}) = \{\nabla f(\bar{x})\}$ and $T_g(\bar{x}) = \{\nabla g(\bar{x})\}$, our results will be true when the underlying functions are C^1.
– If f and g are locally Lipschitzian with $T_f(\bar{x}) = \partial f(\bar{x})$ and $T_g(\bar{x}) = \partial g(\bar{x})$, respectively, our characterizations will be valid and applicable to GCP(f,g) when the underlying data are locally Lipschitzian.
– If f and g are semismooth (in particular, piecewise affine or piecewise smooth) with the Bouligand subdifferential $T_f(\bar{x}) = \partial_B f(\bar{x})$ and $T_g(\bar{x}) = \partial_B g(\bar{x})$, respectively, our characterizations will be valid and applicable to GCP(f,g) when the underlying data are semismooth.

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References

[1] F.H. Clarke, Optimization and Nonsmooth Analysis. Wiley, New York, 1983 (reprinted by SIAM , Philadelphia, PA, 1990).
[2] J.-S. Chen, *The semismooth-related properties of a merit function and a descent method for the nonlinear complementarity problem*, J. Glob. Optim. 2006 (36), pp: 565580.
[3] Jein-Shan Chen and Shao-hua Pan, *A family of NCP functions and a descent method for the nonlinear complementarity problems*, Computational Optimization and Application, 2008(40), pp: 389-404.
[4] B. Chen, X. Chen and C. Kanzow, A Penalized Fischer-Burmeister NCP-Function: Theoretical Investigation and Numerical Results. Mathematical Programming. 88: 211-216 (2000).
[5] R.W. Cottle, J.-S. Pang and R.E. Stone, The Linear Complementarity Problem. Academic Press, Boston, 1992.
[6] M. Fiedler and V. Pták, Some Generalizations of Positive Definiteness and Monotonicity. Numer. Math. 9: 163172 (1966).

[7] D. Gale and H. Nikaid, The Jacobian Matrix and Global Univalence of Mappings. Math. Ann. 159: 8193 (1965).
[8] M.S. Gowda, Inverse and Implicit Function Theorems for *H*-Differentiable and Semismooth Functions. Optimization Methods and Software.19: 443-461 (2004).
[9] M.S. Gowda and G. Ravindran, Algebraic Univalence Theorems for Nonsmooth Functions. Journal of Mathematical Analysis and Applications. 252: 917-935 (2000).
[10] M. S. Gowda and M. A. Tawhid, Existence and Limiting Behavior of Trajectories Associated with P0-Equations. Computational Optimization and Applications. 12: 229-251, (1999).
[11] S. Karamardian, The Complementarity Problem. Mathematical Programming. 2: 107-12 (1972).
[12] R. Mifflin, Semismooth and Semiconvex Functions in Constrained Optimization. SIAM Journal on Control and Optimization. 15: 952-972 (1977).
[13] J.J. Moré and W.C. Rheinboldt, On P- and S- Functions and Related Classes of N-Dimensional Nonlinear Mappings. Linear Algebra and its Applications. 6: 45-68 (1973).
[14] L. Qi, Convergence Analysis of Some Algorithms for Solving Nonsmooth Equations. Mathematics of Operations Research. 18: 227-244 (1993).
[15] L. Qi, C-Differentiability, C-Differential Operators and Generalized Newton Methods. Research Report, School of Mathematics, The University of New South Wales, Sydney, New South Wales 2052, Australia, January 1996.
[16] L. Qi and J. Sun, A Nonsmooth Version of Newton's Method. Mathematical Programming. 58: 353-367 (1993).
[17] R.T. Rockafellar, Convex Analysis. Princeton University Press, Princeton, 1970.
[18] R.T. Rockafellar and R.J.-B. Wets, Variational Analysis. Grundlehren der Mathematischen Wissenschaften, 317, Springer-Verlag, Berlin, 1998.
[19] Y. Song, M.S. Gowda and G. Ravindran, On Characterizations of $\mathbf{P}$- and $\mathbf{P}_0$-Properties in Nonsmooth Functions. Mathematics of Operations Research. 25: 400-408 (2000).
[20] M. A. Tawhid, On Characterization of $\mathbf{E}$- and $\mathbf{E}_0$-Properties in Nonsmooth Functions. Applied Mathematics and Computation. 175: 2: pp. 1609-1618, April 15, (2006).
[21] M. A. Tawhid, Solution of Nonsmooth Generalized Complementarity Problems. Journal of the Operations Research Society of Japan. 54:1, pp. 12-24, (2011).
[22] M.A. Tawhid, On the Local Uniqueness of Solutions of Variational Inequalities under *H*-Differentiability. Journal of Optimization Theory and Applications. 113: 149-164 (2002).
[23] M. A. Tawhid and J-L. Goffin, On Minimizing Some Merit Functions for Nonlinear Complementarity Problems under H-Differentiability, Journal of Optimization Theory and Applications, 139: 127–140 (2008).
[24] M.A. Tawhid and M.S. Gowda, On Two Applications of *H*-Differentiability to Optimization and Complementarity Problems. Computational Optimization and Applications. 17: 279-299 (2000).
[25] M. A. Tawhid, Nonsmooth Generalized Complementarity as Unconstrained Optimization. Journal of Industrial and Management Optimization, 6: 411-423 (2010).
[26] M. A. Tawhid, Unconstrained Methods for Nonsmooth Nonlinear Complementarity Problems. Advanced Modeling and Optimization, 12: 1: pp. 2035, (2010).
[27] M. A. Tawhid, Locally Unique Solutions of Nonsmooth General Variational Inequalities. Nonlinear Analysis: Hybrid Systems, 4: 3, pp. 574-582, (2010).

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